ANALYSIS AND DESIGN OF GENERALIZED MICROSTRIP

By

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KEY TO SYMBOLS

Space Plane Parameters

a	The half width of the narrow conductor in space coordinates
b	The half-width of the wide conductor in space coordinates
h	The half-separation of the conductors in space coordinates.
x	The z-plane abscissa
y	The z-plane ordinate
$z = x + iy$	A general point in space plane coordinates

Normalized Space Plane Parameters

a/h	The ratio of narrow conductor width to conductor separation Eq (A 98)
b/h	The ratio of wide conductor width to conductor separation Eq (A-101)
$X = x/h$	The z-plane abscissa normalized to conductor half separation Eq (A-64)
$Y = y/h$	The z-plane ordinate normalized to conductor half-separation Eq (A-65)

Flux Potential Plane Parameters

a'	The half width of the inner face of the narrow conductor Eq (A 97)
b'	The half width of the inner face of the wide conductor Eq (A-100)
C_p	The polarization capacitance of microstrip on a dielectric sheet Eq (B-11)

c_0	The capacitance of generalized microstrip in vacuum Eq (A-11)
$C0$	A constant connecting the w and z-planes for balanced microstrip Eq (A-25)
$C1$	A constant connecting the w and z-planes for generalized microstrip Eqs.(A-48) and (A-49)
$C2$	A constant connecting the w and z-planes for generalized microstrip Eqs (A-48) and (A-49)
u	The w-plane abscissa
u_0	The flux-potential image of infinity in space coordinates Eq (A-110)
v	The w-plane ordinate
$w = u + iv$	A general point in flux-potential coordinates
Z	The impedance of generalized microstrip on a dielectric sheet. Eq (3-27)
Z_0	The impedance of generalized microstrip in vacuum Eq (A-12)

Normalized Flux-Potential Plane Parameters

$A' = a'/K'$	Normalized half width of the inner face of the narrow conductor
$B' = b'/K$	Normalized half-width of the inner face of the wide conductor
$CPMAX$	Maximum parallel dielectric capacitance = dielectric area divided by c_0 Eq (3-16)
C_e	External capacitance of generalized microstrip Eq (3-21)
C_i	Internal capacitance of generalized microstrip Eq (3-17)
CP	True parallel dielectric capacitance Eq (3-14)
U	u/K
$U0$	u_0/K

v'	v/K'

Electromagnetic Parameters

c	Velocity of light in free-space $\approx 2.9979\times10^{8}$ meters/sec
$\boldsymbol{D}$	The flux, or displacement vector
$\boldsymbol{E}$	The electric field vector
$\boldsymbol{P}$	The polarization field vector
Q	Electric charge.
$U0 = u_0/K$	The normalized flux-potential image of infinity in space coordinates
V	Potential (volts)
v_p	The relative phase velocity
$\epsilon = \epsilon_0\epsilon_r$	The dielectric constant of a lossless dielectric
ϵ_{eff}	The effective relative dielectric constant of microstrip Eq. (3-23)
ϵ_r	The relative permittivity of a dielectric Eq. (3-6)
ϵ_0	The permittivity of free space $= 8.854\times10^{-12}$ Farad/meter
μ_0	The permeability of free space $\approx 1.257\times10^{-6}$ Henry/meter
η	The intrinsic impedance of gree space $= 376.7$ Ohm.
σ	The surface charge density Coulombs/meter^3.

Elliptic Integrals and Functions

$\mathrm{am}(u,k) = \mathrm{am}\ u$	Amplitude u
$\mathrm{am}(v,k') = \mathrm{am}\ v$	Amplitude v
$\mathrm{cn}(u,k) = \mathrm{cn}\ u$	Cosine amplitude u, Jacobian elliptic function
$\mathrm{cn}(v,k') = \mathrm{cn}\ v$	Cosine amplitude v, Jacobian elliptic function.

$\mathrm{dn}(u,k) = \mathrm{dn}\ u$	Delta amplitude u; Jacobian elliptic function
$\mathrm{dn}(v,k') = \mathrm{dn}\ v$	Delta amplitude v; Jacobian elliptic function.
$E(\varphi,k) = E(u)$	Legendre's incomplete elliptic integral of the second kind ($\varphi = \mathrm{am}\ u$)
$E'(\varphi,k) = E(\varphi,k')$	Legendre's associated elliptic integral of the second kind
$E(k) = E = E(\pi/2,k)$	Complete elliptic integral of the second kind
$E' = E(k')$	Associated complete elliptic integral of the second kind
$F(\varphi,k) = u$	Incomplete elliptic integral of the first kind.
k	The modulus of elliptic integrals and functions
$k' = \sqrt{1 - k^2}$	The complementary modulus of elliptic integrals and functions
$K(k) = K = F(\varphi,k)$	Complete elliptic integral of the first kind
$K' = K(k')$	Associated complete elliptic integral of the first kind
$q = e^{-\pi K'/K}$	The nome.
$\mathrm{sn}(u,k) = \mathrm{sn}\ u$	Sine amplitude u: Jacobian elliptic function
$\mathrm{sn}(v,k') = \mathrm{sn}\ v$	Sine amplitude v; Jacobian elliptic function.
$\mathrm{tn}(u,k) = \mathrm{sn}\ u/\mathrm{cn}\ u$	
$\mathrm{tn}(v,k') = \mathrm{sn}\ v/\mathrm{cn}\ v$	
$\Pi(\varphi,\alpha^2,k)$	Incomplete elliptic integral of the third kind

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This dissertation describes a procedure for arriving at a close approximation to the capacitance between symmetrically placed conducting strips, possibly of different widths, on opposite sides of a dielectric sheet. The procedure is based on static methods, following previous work by Black and Higgins for total capacitance of the structure with vacuum everywhere, and Wheeler's method for determining the series component of dielectric capacitance.

Dielectric polarization is included. Refraction at the vacuum/dielectric boundary is ignored in the derived method, but its effect is subsequently shown to be small.

The derived equations are valid for all finite impedance, all values of dielectric constant, and all conductor widths. The maximum absolute error of this static solution is estimated to be $O(0.001\ Z)$, where Z is the characteristic impedance of the generalized microstrip on a dielectric sheet.

The methods described have general application to open transmission lines on a dielectric sheet, for which the appropriate conformal transformations can be found.

The derived equations are implemented in Fortran demonstration programs and subroutines for the basic calculations of generalized microstrip. Tabulated and graphical results are included. Source code listings of the demonstration programs are included in the dissertation.

CHAPTER 1
INTRODUCTION

Description and Classification of Microstrip

The term "microstrip" is an acronym, of unknown authorship, derived from ***micro***wave ***strip*** transmission line. It has come to refer to open strip transmission line, as opposed to closed sandwich strip transmission line, commonly called "stripline." A transverse cross-sectional view of the conventional form of microstrip transmission line (unbalanced) is shown in Fig. 1-1. Until about 25 years ago microstrip was of minor importance in microwave technology. It was little used and, when it was, designs were usually based on simple approximations and experiments. The real need for microstrip came with the development of semiconductors, where the simple configuration of microstrip matched the need for very small transmission lines within semiconductor devices. In the early days of semiconductors frequencies were low and devices relatively simple, requiring little attention to the quality of their connections beyond providing a reasonably good electrical circuit. Driven by recent developments in microwave semiconductor and system technologies, *e.g.*, Microwave Integrated Circuits (MIC's) and Monolithic Microwave Integrated Circuits (MMIC's), and in digital semiconductor speed and complexity, this early indifference to microstrip has changed to intense interest.

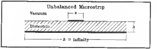


Fig 1-1 Conventional (unbalanced) microstrip

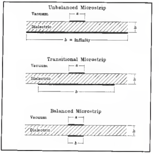


Fig 1-2 Generalized microstrip includes all cases

Microstrip, stripline, coaxial and two wire lines all belong to a class called TEM (Transverse Electric Magnetic) transmission lines. TEM transmission lines have a number of desirable properties: they are compact, broadband and structurally simple. The ground plane and dielectric sheet of unbalanced microstrip are assumed to be of infinite width. In practical terms, they are simply very much wider than the separation h. A distinguishing feature of microstrip is its structural simplicity. Another feature that may not be as obvious is the extreme difficulty of analytical microstrip analysis and, in particular, analysis of microstrip on a dielectric sheet. The principal transmission line parameter is characteristic impedance (Z), while the structural parameters are conductor and dielectric dimensions. For TEM transmission lines, the characteristic impedance is fully determined by the capacitance per unit length which in turn is obtained from the structural parameters. At this writing, the analytical difficulties presented by microstrip are still far from resolved, making microstrip a priority research topic in both microwave and semiconductor technologies. It is the accurate determination of capacitance from microstrip dimensions that presents the major obstacle. The derivation of a close analytic approximation for the capacitance of generalized microstrip on a dielectric sheet will occupy a major part of this dissertation and is one of its principal contributions.

Generalized Microstrip

Generalized microstrip can be divided into three categories as shown in Fig. 1.2. The asymmetrical case, shown at the top, is called "unbalanced" microstrip. The symmetrical case, shown at the bottom, with equal conductor widths, is called "balanced" microstrip. Only two dimensions, conductor width and conductor spacing, are required to describe these cases. In fact, either case can be obtained from the other by the method of images. Between these limiting cases are an infinity of microstrip configurations where both conductors are of finite but unequal width. This can be called transitional microstrip since it provides a transition between the limiting cases. Descriptions of transitional microstrip requires three dimensions, the two distinct conductor widths and their separation. Discussion of generalized microstrip in terms of three discrete cases is for purposes of illustration only, the three cases are totally unified by the derived equations presented in this dissertation.

Unbalanced microstrip is the most common and universally used type of microstrip. Many devices such as antennas and mixers, however, require a balanced input which is not easily obtained in unbalanced microstrip. Generalized microstrip provides a connection between unbalanced and balanced microstrip, enabling direct transformation between the two. Such transformation structures are called balun transformers or simply baluns. These devices occupy an important place among microwave components. Development of precise design methods for baluns is one of the central objectives of this dissertation.

Microstrip Analysis

An important objective of microstrip analysis is to provide accurate design equations to support a variety of microwave and semiconductor devices. Ideally this might consist of a three-dimensional analytic wave solution, yielding the microstrip dimensions and characteristic impedance for generalized conductors on a dielectric sheet Unfortunately this ideal corresponds poorly with the realities of microstrip analysis. Even if a formally exact, analytic wave solution could be found, and it probably could not be, it would almost certainly require purely numerical evaluation Numerical wave solutions have been obtained using finite difference integral equation and Fourier methods These methods are usually computationally intensive and ultimately require comparison to an analytic solution to establish their validity and accuracy

It is an objective of this dissertation to satisfy as many of the characteristics of the ideal solution for microstrip as possible An analytic, closed-form solution is sought, since its accuracy is can be directly established by reference to mathematical and physical theorems

The Two-Dimensional Laplace Equation

Microstrip in a lossless, homogeneous dielectric with lossless conductors propagates a purely TEM wave In this mode there is no axial electric or magnetic field, and the two-dimensional Laplace (static) solution, in the transverse plane, satisfies the one-dimensional wave equation (Telegrapher's equation) in the axial direction For microstrip on a dielectric sheet the dielectric is inhomogeneous, with the flux partially in dielectric and partially in free-space In general, the flux impinging on this boundary is distorted, the exception being flux that is normally incident to the dielectric boundary This distortion, called refraction, is a result of the dielectric boundary conditions Restrictions of the Schwarz-Christoffel transformation are such that the general boundary conditions cannot be imposed, a solution, however, can be obtained by neglecting refraction This is equivalent to assuming normal incidence for all flux impingent on the dielectric/free-space boundary This is the only analytic assumption required to obtain a static solution The effect of this assumption is assumed to be small and will be shown to be so by comparison to a numerical solution that includes the effects of refraction

The solution derived in this dissertation consists of the exact solution for the static capacitance with vacuum everywhere and a second exact solution for the polarization capacitance of the dielectric, both obtained with the Schwarz-Christoffel transformation The mixed-dielectric equations, obtained by combining these two solutions, are analytically exact except for refraction effects

Assumptions

Use of the static solution as a wave solution requires the assumption that the microstrip propagates a purely TEM wave For microstrip in a homogeneous dielectric this assumption is precisely true; when only the separating sheet is dielectric, however, it is an approximation This well known approximation is commonly called the quasi TEM or quasi-static assumption and assumes that the axial electric and magnetic fields are negligible

Losses in the conductors and dielectric give rise to an axial potential gradient and thus an axial electric field so the conductors and dielectric must be assumed to be lossless This is a good assumption for silver, copper and even gold conductors   Similarly, for microwave materials, the lossless assumption for the dielectric is a good one. The dielectric is also assumed to be nonmagnetic, a property of most microwave dielectric materials

Use of the Schwarz Christoffel (conformal) transformation requires two additional assumptions: (1) the effect of refraction at the dielectric/free-space boundary is assumed to be negligible and (2) the conductors are assumed to be so thin as to approximate mathematical planes   The first is necessary because conformal mapping cannot treat general dielectric boundary conditions   The second is to simplify the Schwarz Christoffel transformation sufficiently to yield a closed-form analytic solution

The quasi-TEM solution for the inhomogeneous dielectric case does not display the frequency dependence (dispersion) which is known to exist in microstrip, meaning that the assumptions are imperfect   They are, however, quite good, and using reasonable design restrictions so as not to significantly violate the assumptions, results have been excellent   The solution is exact for $\epsilon_r = 1$ and $\epsilon_r \to \infty$ and, with the exception of refraction error, is also exact, at the static limit, for all other values of relative dielectric constant.

Related Prior Art

Conformal Mapping Solutions

The most closely related prior art is the work of this author [1], concerned with generalized microstrip on a dielectric sheet   The homogeneous dielectric case of generalized microstrip was previously analyzed by Black and Higgins [2], using exact conformal mapping. A modification of their analysis is used in this dissertation to determine the total capacitance of generalized microstrip in free-space.

The special case of a conducting strip separated from an infinite conducting plane by a dielectric sheet was analyzed and characterized by Dukes [3-4] with the aid of an electrolytic tank, and later by Wheeler [5-7] using approximate conformal mapping and an interpolation technique which produces a correction term to the result obtained by Dukes [3].

In a 1984 paper, Callarotti and Gallo [8] describe a method for analysis of unbalanced microstrip, using conformal mapping to obtain the dielectric boundary in flux-potential coordinates, followed by a numerical evaluation of microstrip capacitance in the transform plane.

Dielectric Green's Function

Weiss *et al.* [9-10] developed computer programs treating, as one option, the case of unbalanced microstrip on a dielectric sheet using a numerical method, the dielectric Green's function as described by Sylvester [11]. Like the references above, this is a quasi-TEM solution but differs in that refraction effects are included. It is used in Chapter 4 to estimate the error introduced by the omission of refraction in the analytic solution. None of its numerical features are used in the solution obtained in this dissertation.

Balun Transformers

The tapered microstrip balun configuration was described by Rumsey [12]. Baluns of this type can simultaneously provide both unbalanced to balanced mode conversion and a suitable impedance transformation between an unbalanced coaxial transmission line and the balanced terminals of an antenna or other device. The Klopfenstein [13] impedance taper provides an optimum impedance transformation but requires a detailed knowledge and control of impedance at every point along the taper. The procedures provided in this dissertation are intended to accurately provide those parameters.

Form of the Solution

This dissertation is concerned with generalized microstrip of the type analyzed by Black and Higgins [2], but on a dielectric sheet as treated, for special cases, by Dukes [3-4], and Wheeler [5-7]. Exact conformal mapping is used, as opposed to approximate conformal mapping employed by Wheeler [5]. Disadvantages of this approach are the added complexity of the exact solution and the special functions and integrals which arise. These disadvantages are more than offset by the lack of approximation in the resulting solution.

For a given characteristic impedance Z, and relative dielectric constant ϵ_r, a parameter R determines the ratios of conductor width to separation, $a/h(R,Z,\epsilon_r)$ and $b/h(R,Z,\epsilon_r)$. With $R = 0.0$, the conductors are of finite and equal width, while $R = 1.0$ gives a single, finite conducting strip over an infinite ground plane. Values of R between these limits result in two finite conducting strips of unequal width.

It would be desirable to have explicit expressions of the form $R = R(a/h, Z, \epsilon_r)$ and $R = R(b/h, Z, \epsilon_r)$. It is, however, an unfortunate consequence of the solution that R arises as the parameter of an elliptic integral of the third kind and cannot be expressed in this way. The effect of this implicit, rather than explicit, relationship between conductor width and the parameter R is a somewhat more difficult calculation than would be the case if an explicit relationship could be found.

Organization and Scope

The intended audience for this dissertation includes microwave design engineers and researchers concerned with electromagnetic phenomena. The main body is devoted primarily to development of the physical aspects of generalized microstrip analysis and balun design. Mathematical developments, such as conformal mapping, are treated in detail in the Appendixes. Source listings for computer programs are also listed in the Appendixes. This organization is not intended to diminish the importance of either the conformal derivations or the computer programs, but rather to avoid mixing the physical and mathematical developments in the interest of clarity.

Of general interest to the microwave and semiconductor communities are the design equations for unbalanced and balanced microstrip in their most accurate form. With an estimated accuracy of 0.001 of impedance, these equations are thought to yield the most accurate quasi-TEM values of microstrip parameters available. These equations provide a standard for testing numerical microstrip analysis programs and assessing their accuracy at static limits as well as for direct use in the design of unbalanced and balanced microstrip

CHAPTER 2
DESCRIPTION OF THE CONFORMAL TRANSFORMATIONS

Background

Conformal mapping, and the Schwarz-Christoffel transformation in particular, provide a formalism for transforming a relatively complicated geometry, such as that of generalized microstrip, into a simpler geometry where solutions to the two-dimensional Laplace equation can be more easily obtained, usually by inspection. The resulting solutions apply to a transverse cross-section of the structure, with the microstrip assumed to be constant in the axial direction. Direct application of conformal mapping is restricted to cases with homogeneous dielectric. Two conformal transformations are used in the analysis of generalized microstrip; both are for homogeneous dielectric. The first, derived in Appendix A, is used to obtain the total capacitance of the conducting strips with vacuum everywhere, following Black and Higgins [2]. The second conformal transformation, derived in Appendix B, is used to find the polarization capacitance of the dielectric due to electrification of the inner faces of the conducting strips, i.e., the faces in direct contact with the dielectric sheet. This is also a homogeneous case since the polarization field is entirely within the dielectric

The Free-Space Transformation

The Schwarz-Christoffel transformation conformally transforms the interior of a polygon into a half plane, conformal meaning that angles are preserved across the transformation. In space coordinates, Fig. A-1 of Appendix A, the Schwarz-Christoffel polygon is the upper half of the microstrip; it encloses the conductors and all electric field lines originating or terminating on them. The space coordinate (z-plane) polygon thus includes infinity as one of its vertices. The free-space w to z plane transformation is obtained using two sequential conformal transformations: the first transforms the z plane polygon into the upper half of the t plane while the second transforms the upper half of the t plane into the w-plane, or flux-potential coordinates. All three are complex planes with $w = u + iv$ and $z = x + iy$, while only the real axis of the t plane is explicitly used. The conductors in space coordinates are transformed to the w-plane in such a way that both their inner and outer surfaces face the interior. The transformed flux lines, in the w-plane, are lines of constant v and the equipotential lines are those of constant u. When the z-plane is mapped into a square, as in Fig. A-1, the squares in flux-potential coordinates map into conformal squares in space coordinates. This is shown in Figs. 2-1 and 2-2, where each square in Fig. 2-2 maps to a conformal square in Fig. 2-1. Presentation of the w-plane as a square is done to illustrate conformality and is not generally representative of its actual dimensions.

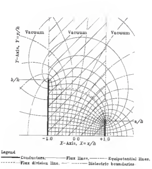


Fig. 2-1 Generalized microstrip for $R = 0.6$, and free-space impedance 188.4 Ohms, space coordinates.

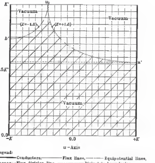


Fig 2-2 Generalized microstrip for $R = 0.6$, and free-space impedance 188.4 Ohms, flux-potential coordinates

Space and Flux-Potential Coordinates

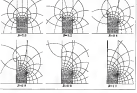
While it is not necessary to fully understand the conformal mapping derivations of Appendixes A and B to follow the development of the capacitance and impedance equations, it is necessary to understand the nature of the transformed (flux-potential) parameters and their relationship to generalized microstrip in space coordinates. Figure A-1 and Table A-1 in Appendix A show the relationship between points in space coordinates and their images in flux-potential coordinates. These relationships are also illustrated in Figs. 2-1 and 2-2, which show the flux lines, equipotential lines and dielectric boundaries. Parameter definitions for both the space and flux potential planes and, where appropriate, the equation number for calculation of the parameter are given in the List of Symbols.

Space coordinates

Figure 2-1 shows the upper half-plane of generalized microstrip with unequal width conductors. The space coordinate plane is normalized by the half-separation of the conductors, such that $X = x/h$ and $Y = y/h$. The conductors are the bold vertical lines at $X = -1.0$ and $X = +1.0$. They are assumed to be at potentials −1 V and +1 V, respectively. The thin solid lines between the conductors are the flux lines. The curved short-dashed line, terminating at the upper edge of the narrow conductor and on the inner face of the wide strip, is the flux division line. All flux below this line terminates on the inner face of the narrow conductor and all flux above terminates on the outer face of the narrow conductor. The alternate long-short dashed lines, orthogonal to the flux lines, are the lines of equal potential. The shaded region between the conductors indicates the area that will be occupied by dielectric but is currently filled with vacuum. The vertical long-double-short dashed lines, bounding the shaded area above the conductors, are the nonmetallized surfaces of the infinitely wide dielectric sheet separating the conductors. In the drawing the shaded area is filled (as indicated) by vacuum and the flux lines shown are those appropriate to vacuum.

Flux-potential coordinates

Fig. 2-2 is the flux-potential image of the microstrip structure, in Fig. 2.1, resulting from the conformal transformation derived in Appendix A. The line coding and other conventions are the same as in space coordinates. Three points in flux-potential coordinates are of special significance. These are at ($u=K$, $v=a'$), ($u=-K$, $v=b'$) and ($u=u_0$, $v=K'$). Point (K, a') is the flux potential image of point ($X=+1.0$, $Y=a/h$) in space coordinates. The conductor segment at $u = +K$ extending from $v = 0.0$ to $v = a'$, is the flux-potential image of the inner face of the narrow conducting strip in space coordinates. The conductor segment above point (K, a'), and extending to point (K, K'), is the flux potential image of the outer face of the narrow conducting strip in space coordinates. Point ($-K$, b') has the same meaning for the wide conducting strip. Point (u_0, K') is the flux-potential image of infinity in space coordinates. The curved long-double-short dashed lines, labeled $X = +1.0$ and $X = -1.0$, are the flux potential images of the nonmetallized surfaces of the infinite-width dielectric sheet separating the conductors. The dielectric boundaries are transformed between space and



Legend:
——— Conductors, ——— Flux lines, — — Equipotential lines,
- - - - Dielectric boundaries

Fig. 3-3 Generalized microstrip for B = 0.0 to 1.0, and free-space impedance 94.2 Ohms, space coordinates

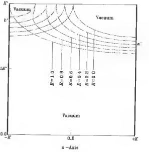


Fig. 2-4 Generalized microstrip for $R = 0.0$ to 1.0, and free-space impedance 94.2 Ohms, flux-potential coordinates

flux-potential coordinates using eqs. (A-64) and (A-65) for the general case. Equations (A-71) and (A-75) can be used for the special case of balanced microstrip. The shaded regions in both figures are filled with vacuum but represent the areas that will be filled with dielectric. The flux and equipotential lines shown are those appropriate to free-space.

The solid horizontal lines in flux potential coordinates are flux lines, while the vertical alternate long-short dashed lines are the lines of equal potential. The horizontal short-dashed line at $v = a'$ is the division line between flux from the inner and outer faces. Flux below this line is from the inner face, while that above is from the outer face. Limits of the parameter a' are, $K'/2 \leq a' \leq K'$. Thus at least half of the electric field is always between the inner faces of the conductors. All flux from the outer face is partially in the unshaded (free-space) region and partially in the shaded (dielectric) region whereas flux from the inner face is entirely within the shaded region.

Variation of the Fields and Dielectric Boundaries with R

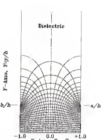
Figures 2-3 and 2-4 show generalized microstrip for six values of R from 0.0 to 1.0 in space and flux-potential coordinates. The flux and equipotential lines have been omitted in Fig. 2-4 flux-potential coordinates for clarity and readability of the composite graph. All six illustrations are at the same impedance, 94.2 Ω. Both conductors simultaneously change width as R is varied to maintain this constant impedance. The dielectric boundaries in flux-potential coordinates are labeled with the appropriate values of R to allow correlation with their space coordinate images. The lines and boundaries are coded as in Fig. 2-1. Figures 2-3 and 2-4 illustrate the changes in the field lines and dielectric boundaries as strip width is changed, at constant impedance, by variation of *R*.

Effect of the Dielectric Sheet

The introduction of dielectric with relative permittivity greater than one into the shaded regions has two effects. The more important effect is that the dielectric is polarized by the electric field, with the polarization capacitance rapidly becoming the dominant capacitance as the relative dielectric constant is increased. The second effect is refraction at the vacuum/dielectric boundary. This effect, of necessity, is not included in the calculations but its effect is estimated in Chapter 4.

All illustrations of fields in this dissertation are either vacuum everywhere or of the polarization field within the dielectric. Both of these are homogeneous cases. No illustrations are provided for the inhomogeneous case. This omission is necessary because the fields cannot be explicitly calculated for the inhomogeneous case by the methods of this dissertation.

Figure 2-5 shows the polarization field and equipotential lines within the dielectric sheet in the upper half plane of space coordinates. The relationships that determine this field distribution are derived in Appendix B. The case illustrated is for balanced microstrip. The figure shows twenty-one field lines, equally spaced; hence, 95 percent of the field is below the uppermost field line and only 5 percent lies above it. The dielectric boundaries are also the outermost field lines; these extend to infinity. The concentration of electric field at the conductor edges



Legend:
——— Conductors, ——— Flux lines, — — Equipotential lines.

Fig. 2-5 The polarization field in dielectric for equal-width conducting strips, $E = 0.0$, in space coordinates.

indicates the accumulation of charge at those points. Polarization is present, only for relative permittivity greater than one, *i.e.*, only when a polarizable dielectric is present.

For a relative permittivity of one (free space) the electric field terminating on the inner faces of both conductors is that field below the flux division line at a', in Fig. 2-2, and with no polarization field. With infinite permittivity the electric field terminating on the inner faces of both conductors is that shown in Fig. 2-5. Between these limits the field on the inner faces is a mixture of the free-space and polarization fields.

CHAPTER 3
DERIVATION OF EFFECTIVE DIELECTRIC CONSTANT

Definitions and Fundamental Relationships

Conformal mapping analysis of generalized microstrip, and in particular the inhomogeneous dielectric case, involves specialized techniques and relationships that are not often encountered elsewhere. This chapter is intended to introduce those definitions, relationships and techniques required for a full understanding of the derivations unique to microstrip analysis by conformal mapping.

Microstrip Capacitance Calculations

The principal reward for the effort involved in deriving the conformal transformation of generalized microstrip is the ease with which capacitance can be calculated in the flux potential plane. Since the electric field is uniform in flux-potential coordinates, with homogeneous dielectric, capacitance is equal to dielectric constant times conductor width divided by conductor separation. This relationship is well known and widely used but in the interest of completeness and because of its importance within this dissertation, its derivation with respect to microstrip is included here.

The w-plane transformation of microstrip derived in Appendix A and discussed in the previous chapter has conductors at potentials V_a and V_b, which gives rise to an electric field of magnitude $\boldsymbol{E}$. From Gauss' law $\boldsymbol{D} = \epsilon\boldsymbol{E} = \sigma$ and

$$E = \sigma/\epsilon, \quad \text{Volts/meter} \tag{3-1}$$

where σ is the surface charge density on the inner surface of one of the conductors. Since the electric field is constant in magnitude and direction, the potential difference between the conductors is given by

$$V_b - V_a = Es \quad \text{Volts} \tag{3-2}$$

where s is the separation of the conductors. The total charge per meter, Q, on either plate is given by $Q = \sigma \times area = \sigma \times w \times (1 \text{ meter}) = \sigma w$, where w is the conductor width. Substitution of ϵE for σ gives

$$Q = \epsilon E w \quad \text{Coulombs/meter} \tag{3-3}$$

Capacitance is defined as total charge on either conductor divided by potential difference of the conductors. Dividing (3-3) by (3-2) gives

$$c = Q/(V_b - V_a) = \epsilon_0 \epsilon_r w/s \quad \text{Farads/meter} \tag{3-4}$$

From (3-4) it is seen that capacitance in the w-plane is calculated as dielectric constant times conductor width divided by conductor separation. The above derivation is for an arbitrary real dielectric constant, $\epsilon = \epsilon_0 \epsilon_r$. For free-space, $\epsilon_r = 1$ giving

$$c_0 = \epsilon_0 w/s \quad \text{Farads} \tag{3-5}$$

Relative Dielectric Constant.

If the geometric capacitance of a particular capacitor is c_0 when filled with vacuum, and c when a capacitor with the same geometric capacitance is filled with a dielectric substance, the relative dielectric constant can be obtained by dividing (3-4) by (3-5) giving

$$\epsilon_r = \frac{c}{c_0} \tag{3-6}$$

The concept of geometric capacitance is based on an ideal capacitor whose free-space capacitance per unit length is completely characterized by (3-5). The division of (3-5) by (3-4), which results in (3-6) can be viewed as a normalization process with normalization constant $c_0 = \epsilon_0 w/s$. Since this normalization constant contains both the dimensions and the free-space dielectric constant, the result, ϵ_r, can be interpreted as the pseudo-capacitance of a unit capacitor with $w = s = 1$, whose capacitance is dependent only on ϵ_r. This interpretation is of little importance with respect to relative dielectric constant since the unit pseudo-capacitor is uniformly filled with a single dielectric substance. The utility of the concept is in the calculation of effective dielectric constant where the unit capacitor is partially filled with a dielectric substance and partially with vacuum.

Effective Dielectric Constant.

If an ideal capacitor is only partially filled with a dielectric substance and partially with vacuum, as is the case with the flux-potential plane representation of microstrip on a dielectric sheet, the definition for relative dielectric constant can still be used if it is understood to represent effective dielectric constant. Effective dielectric constant is, therefore, the single relative dielectric constant filling the entire flux-potential plane that produces an equivalent capacitance. Effective dielectric constant can be written

$$\epsilon_{eff} = \frac{c_{eff}}{c_0} \tag{3-7}$$

where C_{eff} is the effective capacitance with space partially filled with dielectric and C_0 is the free-space capacitance of the microstrip. For application of this concept to the flux/potential representation of microstrip, the flux/potential plane is divided into subregions, each entirely filled with either dielectric or vacuum, and with their dielectric/vacuum boundaries aligned either with flux or equipotential lines. This latter requirement eliminates refraction effects which are not permissible in conformal maps. Each of these subregions is then normalized to C_0 and each represents an appropriate fraction of a unit capacitor, with the sum of the areas of the subregions equal to unity. Subregions with their dielectric/vacuum boundaries aligned with flux lines are in parallel while those with boundaries aligned with equipotential lines are in series. Each subregion is a capacitor with homogeneous dielectric whose pseudo-capacitance is computed as conductor width over conductor separation times either ϵ_r or unity, depending on whether the subregion is filled with dielectric or vacuum. The effective dielectric constant is obtained by appropriately combining the pseudo-capacitances of the subregions in series or parallel.

Equivalent Circuit Components

Derivation of effective dielectric constant follows the principles set down in the previous section and is derived with reference to Fig. 2.2. The region above a' is inhomogeneous since the flux from the outer face of the narrow conductor is partially in free-space and partially in dielectric. In the region below a', the flux from the inner faces of the conductors is seen to be entirely in dielectric. The curved dielectric boundaries above h' do not align with either equipotential lines (series component) or flux lines (parallel component) but are a combination of both. Figure 3-1 is an equivalent circuit of microstrip, showing the four capacitive components that must be determined to find the effective dielectric constant. In Fig. 3-1 the two capacitors in series represent the capacitance attributable to flux originating on the outer face of the narrower strip. This will be called the external capacitance. Conversely capacitance due to flux originating on the inner face will be called the internal capacitance. The concept of internal and external capacitances will be further formalized in the following sections.

Fractional Parameters

As outlined earlier in this chapter, the effective dielectric constant is found using fractional pseudo-capacitances of subregions obtained by normalization of the subregion capacitance by the total free-space capacitance. For consistency it will be necessary to define fractional width parameters as well as the fractional subregion capacitance parameters. Fractional width parameters are defined as partial conductor width over total conductor width, just as fractional capacitance is defined as partial capacitance divided by total capacitance. In all cases the capacitances will be "geometric" capacitance, *i.e.*, derived as conductor width over conductor separation, and assumed to be in vacuum.

Fractional Width Parameters

The width of the inner face of the narrow conductor, including both upper and lower half-planes, is

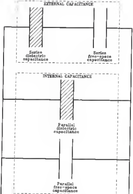

EXTERNAL CAPACITANCE
Series dielectric capacitance
Series free-space capacitance
INTERNAL CAPACITANCE
Parallel dielectric capacitance
Parallel free-space capacitance
−
+

Fig. 3-1 Equivalent circuit in the transverse plane, for generalized microstrip on a dielectric sheet.

$$2a'/2K' = a'/K' \qquad (3\text{-}8)$$

The term a'/K' will occur frequently in the derivation, so let

$$A' = a'/K'$$

Similarly the fractional width of the outer face is

$$1 - A' = 2(K' - a')/2K' = 1 - a'/K' \qquad (3\text{-}9)$$

Fractional Free-Space Capacitance Parameters

The total free-space capacitance, including both upper and lower half planes, is given by the dielectric constant of free-space times conductor width over conductor separation

$$c_0 = \epsilon_0 2K'/2K = \epsilon_0 K'/K$$

The partial parallel free-space capacitance of the inner conductor face of the narrow conductor is

$$\epsilon_0 2a'/2K = \epsilon_0 a'/K \qquad (3\text{-}10)$$

The fractional parallel free-space capacitance of the inner face is

$$A' = (\epsilon_0 a'/K)/(\epsilon_0 K'/K) = a'/K' \qquad (3\text{-}11)$$

The fractional parallel free-space capacitance of the outer face of the narrow conductor is

$$1 - A' = (\epsilon_0 2(K' - a')/2K)/(\epsilon_0 K'/K) = 1 - a'/K \qquad (3\text{-}12)$$

A consequence of normalization is that the fractional widths, capacitances and areas are algebraically equivalent. They are not, however, physically equivalent. In the derivation which follows, the physical usage of the parameters will be indicated at points where confusion might result.

Fractional Parallel Dielectric Polarization Capacitance

The parallel dielectric polarization capacitance is derived in Appendix B and given by (B-11) in unnormalized form, designated c_p. For use as a fractional capacitance it must be normalized giving

$$c_p/c_0 = [(\epsilon_0 2P(p)/P'(p))/(\epsilon_0 K'/K)](\epsilon_r - 1) \qquad (3\text{-}13)$$

Let

$$C^P = (2P(p)/P'(p))/(K'/K), \qquad (3\text{-}14)$$

then

$$c_p/c_0 = C^P(\epsilon_r - 1). \qquad (3\text{-}15)$$

In Fig. 3-1 this is the parallel dielectric capacitance. It is analogous to the parameter designated q'' by Wheeler [6]. Previous to Wheelers' use, Dukes [3] used the parameter implicitly in his analysis of the special case of equal width strips. The value, valid for the special case of equal strips, was obtained as half the capacitance of shielded strip transmission line. The value is not valid for generalized microstrip, but is exactly equivalent with $R = 0.0$ and, by images, $R = 1.0$. To this point, two of the four capacitances of the equivalent circuit have been found, specifically the two parallel capacitances comprising the internal capacitance. It remains to find the series capacitances of the external capacitance.

The Relationship of Capacitance to Area

Area in the flux-potential plane has little meaning when taken alone. Clearly each subregion in the flux-potential plane has an area and the total area of these subregions must be the area of the flux potential plane. Beyond this area alone has little meaning. It is the interpretation of area as a capacitance that is meaningful. The capacitance of an area is indefinite unless either width or separation is specified. In the derivation of effective dielectric constant the area of the dielectric region of the flux-potential plane will be used to find the series dielectric component.

By conservation of matter the total dielectric area in the flux-potential plane is fixed. The curved dielectric boundary, see Fig. 2-2, gives rise to both a parallel component and a series component of dielectric. The series component cannot be directly found. The parallel component, however, is known, it is exactly *CP* given by (3.14) above. If all of the dielectric in Fig. 2-2 is assumed to be in parallel, i.e., with its dielectric/vacuum boundary aligned with flux lines, it results in a parallel dielectric capacitance which will be designated *CPMAX*. It is the maximum possible value of parallel dielectric capacitance with all dielectric in parallel. Meanwhile, the actual extent to which the dielectric participates in the parallel dielectric capacitance is known to be *CP*. By conservation of matter the surplus parallel dielectric, *CPMAX* − *CP*, is the dielectric of the series component.

Calculation of *CPMAX*

The fractional dielectric capacitance *CPMAX* is analogous to the parameter q' in Wheeler [6] which he describes as an area. For reasons given in the previous section, the designation of fractional capacitance is thought to be more physically relevant.

Let a_d be the area of the dielectric in the upper half-plane, then the total dielectric area is $2a_d$. The effective width of the dielectric area is $w = 2a_d/s$, where s = conductor separation = $2K$, giving $w = a_d/K$. The effective parallel free-space capacitance of the of the dielectric region is $\epsilon_0 w/s = \epsilon_0 w/2K = \epsilon_0 a_d/2K^2$. The fractional parallel capacitance of the total dielectric region is

$$CPMAX = (\epsilon_0 a_d/2K^2)/(\epsilon_0 K'/K) = a_d/2KK'.$$

The area, a_d, is obtained by integrating the area under the curved dielectric boundaries in Fig. 2-2. This gives

$$CMAX = \frac{1}{2KK'}\int_{-K}^{+K} v(u)\,du\,, \tag{3-16}$$

where $v(u)$ is the function describing the dielectric. The integrand, $v(u)$, is given by eq. (A-64) for the general case and (A-62) for the case $R = 0.0$. Use of (A-64) requires an iterative procedure. An additional difficulty with the area integration occurs when the dielectric boundary becomes double valued with respect to u, as shown in Fig. 2-4, for $R = 0.8$. When this occurs, special handling is required.

Internal and External Capacitances

The total microstrip capacitance can be represented as two capacitances in parallel, designated C_i for the internal capacitance and C_e for the external capacitance (see Fig. 3-1). these capacitances are defined as follows:

C_i = the capacitance resulting from flux terminating on the inner face of the narrow conductor and

C_o = the capacitance resulting from flux terminating on the outer face of the narrow conductor.

Internal Capacitance

The internal capacitance is composed of two fractional capacitances in parallel. These are the electric component and the dielectric polarization component. The nature of these capacitances is illustrated by reference to Fig. 2-2. The internal free-space capacitance, given by (3-11), is A'. If the strips are immersed in an infinite, homogeneous dielectric with relative dielectric constant $\epsilon_r > 1$, the internal capacitance can be written

$$C_i = A'(\epsilon_r - 1) + A'$$

The first term on the right is the dielectric polarization term and the second is the electric, or free-space, term. For the homogeneous case the polarization and electric fields are coincident everywhere, so the internal capacitance can, equivalently, be written as $C_i = \epsilon_r A'$.

The polarization field of interest in microstrip is that shown in Fig. 2-5. For this case, the polarization field is constrained by the finite dielectric boundaries. The electric and polarization fields are no longer coincident and the internal capacitance must be expressed as the polarization capacitance, given by (3-15), and the free-space capacitance A', giving

$$C_i = CP(\epsilon_r - 1) + A'. \tag{3-17}$$

Equation (3-17) is valid for all values of dielectric constant. For free-space it reduces to $C_i = A'$ as it should.

External Capacitance

The difference $CPMAX - CP$ represents that portion of dielectric that does not participate in the true parallel capacitance derived in Appendix B and, therefore, is assumed to be the series dielectric component. The difference represents an extra conductor width in normalized coordinates with a conductor separation of unity. It can therefore be directly interpreted as a dielectric area.

Figure 3-2 is a schematic representation of the dielectric and free-space areas comprising the internal and external capacitances in parallel. The external capacitance is the series combination of dielectric and vacuum components, of width $1 - A'$, at the top of the schematic. The width is that given by (3-9). The shaded region on the upper left, of separation S and width $W = 1 - A'$, represents the series dielectric component of the external capacitance. The unshaded region with separation $1 - S$ is the series free-space capacitance. With the width of the series dielectric equal to $1 - A'$ and its area, $CPMAX - CP$, the separation, S, can be obtained from

$$W \times S = CPMAX - CP,$$

giving

$$S = (CPMAX - CP)/W \tag{3-18}$$

The inverse of the external capacitance is

$$1/C_e = 1/(\epsilon_r W/S) + 1/(W/(1 - S)) \tag{3-19}$$

giving the external capacitance as

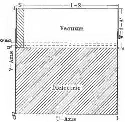


Fig 3-2 Normalized w-plane showing fractional capacitances

$$C_e = \frac{W}{1-(1-\frac{1}{\epsilon_r})S} \tag{3-20}$$

Substitution of $1 - A'$ for W and the right-hand side of (3-18) for S in (3-19) yields the expression for external capacitance

$$C_e = \frac{1-A'}{1-(1-\frac{1}{\epsilon_r})\frac{CPMAX-CP}{1-A'}} \tag{3-21}$$

In the limit $\epsilon_r \to \infty$ the external capacitance reaches a finite limit. This is the maximum possible value of external capacitance.

$$\underset{\epsilon_r \to \infty}{\text{Limit}}\ C_e = \frac{1-A'}{1-\frac{CPMAX-CP}{1-A'}} \tag{3-22}$$

Effective Dielectric Constant

Finally, the effective dielectric constant is obtained by combining the internal capacitance, (3-17), and the external capacitance, (3-21), in parallel giving

$$\epsilon_{eff} = CP\,(\epsilon_r - 1) + A' + \frac{1-A'}{1-(1-\frac{1}{\epsilon_r})\frac{CPMAX-CP}{1-A'}} \tag{3-23}$$

For calculation, substitute a'/K' for A', where a' is given by (A-97) for the general case and (A-104) for the case $R = 0.0$. Compute CP using (3-14) and $CPMAX$ with (3-16).

It is noteworthy that the width of the wide conducting strip does not appear explicitly in the equation for ϵ_{eff}. It does, however, influence the value implicitly through via the calculations for $CPMAX$, CP.

Three derived parameters are involved in the calculation of ϵ_{eff}. They are

$A'(R,Z_0)$ = the fractional parallel free-space capacitance,

$CPMAX(R,Z_0)$ = the maximum value of fractional parallel dielectric capacitance and

$CP(R,Z_0)$ = the true value of fractional parallel dielectric capacitance

Their behavior as functions of free-space impedance, Z_0, and the parameter R is shown in Figs 3-3, 3-4 and 3-5, respectively

Microstrip Impedance

The characteristic impedance of microstrip on a dielectric sheet is completely determined by the effective dielectric constant and the conductor widths provided that the dielectric sheet is non-magnetic, *i e* , the permeability of the dielectric is equal to the permeability of free-space. This is nearly always the case for microwave dielectric materials

The characteristic impedance (Z) of microstrip is defined as

$$Z = \sqrt{\frac{L}{C}}$$

In free-space the characteristic impedance of a microstrip transmission line can be written

$$Z_0 = \sqrt{\frac{\mu_0}{c_0}} \tag{3-24}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Henry/meter, the permeability of free space, and c_0 is the geometric capacitance per meter In the presence of a non magnetic dielectric sheet between the conductors the characteristic impedance can be written

$$Z = \sqrt{\frac{\mu_0}{c_{eff}}} \tag{3-25}$$

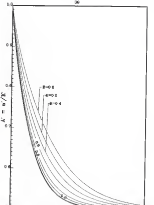


Fig 3-3 Normalized internal free-space capacitance (A') versus free-space impedance and the parameter R

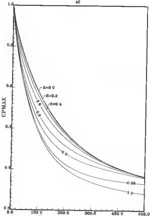


Fig. 3-4 Normalized maximum parallel dielectric capacitance (CPMAX) versus free-space impedance and the parameter R.

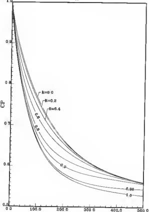


Fig 3-5 Normalized true parallel dielectric capacitance (CP) versus free-space impedance and the parameter K.

Dividing (3-25) by (3-24) cancels μ_0, giving

$$Z = \frac{Z_0}{\sqrt{\dfrac{c_{eff}}{c_0}}} \tag{3-26}$$

where c_0 is the geometric free-space capacitance of the microstrip, given by Eq (A-11) of Appendix A, Z_0 is the free-space characteristic impedance given by Eq (A-12), c_{eff} is the effective microstrip capacitance on a dielectric sheet, and Z is the characteristic impedance of the microstrip in the presence of the dielectric sheet. Since $\epsilon_{eff} = c_{eff}/c_0$, (3-26) can be written

$$Z = \frac{Z_0}{\sqrt{\epsilon_{eff}}} \tag{3-27}$$

Approximation of ϵ_{eff}

The most complex and time consuming procedure in the use of (3-23) is the computation of *CPMAX*, by numerical integration. Wheeler [6] obtained this value by approximating the dielectric boundary with an ellipse. This method gives good accuracy in the mid-range of impedance for $R = 0.0$. Because of the wide range of boundary shapes occurring in generalized microstrip (see Fig. 2-4), this procedure cannot be used in general. The following empirical formula does not require integration and leads to the calculation of the characteristic impedance of generalized microstrip on a dielectric sheet within less than ± 0.25 percent of the value obtained using (3-23):

$$\epsilon_{eff} = CP(\epsilon_r - 1) + A' + \frac{1 - A'}{1 - 0.536\left(1 - \frac{1}{\epsilon_r}\right)\frac{CP - A'}{1 - A'}} \qquad (3\text{-}28)$$

Equation (3-27) is valid to the specified accuracy for $0 < Z_0\,\Omega$, $0 \le R \le 1$ and $1 \le \epsilon_r$. It is the empirically adjusted average of the cases with no dielectric above A' and that where all of the dielectric area $CP - A'$ is in series

CHAPTER 4
COMPARISON TO PREVIOUS RESULTS

Analytical Comparison

The analyses of Dukes [3] and Wheeler [6] are not for generalized microstrip but for the special cases R = 0.0 and R = 1.0. Their formulation of ϵ_{eff} (except for notation) is, however, closely related to that derived in the previous chapter.

To compare the ϵ_{eff} equations derived by Dukes and Wheeler, the external capacitance, eq. (3-21), can be written in the form

$$C_e = (1 - A')\frac{1}{1 - X} \qquad (4\text{-}1)$$

where

$$X = (1 - \frac{1}{\epsilon_r})\frac{CPMAX - CP}{1 - A'}$$

Since X is always less than 1, (4-1) can be expanded in a binomial series, giving

$$C_e = (1 - A')(1 + X + X^2 + X^3 + \ldots) \qquad (4\text{-}2)$$

The effective dielectric constant can be approximated using one or more terms of (4-2) as the external capacitance, in parallel with the internal capacitance, given by eq. (3-17).

Comparison with Dukes

Using only the first term of (4-2) gives

$$\epsilon_{eff} = CP(\epsilon_r - 1) + A' + 1 - A' = CP(\epsilon_r - 1) + 1 \qquad (4\text{-}3)$$

Except for notation, this is precisely the equation derived by Dukes [3]. Effectively, this formulation simply ignores the series dielectric portion of the external capacitance. Z computed using (4-3), and using the exact conformal equations from the appendixes, is accurate to better than 0.75 percent of Z, computed with (3-23). This shows that, at least for the balanced microstrip case, the dielectric in the external region is not very important.

Comparison with Wheeler

Including the first two terms of (4-2) gives

$$\begin{aligned}\epsilon_{eff} &= CP(\epsilon_r - 1) + A' + 1 - A' + (1 - 1/\epsilon_r)(CPMAX - CP)\\ &= CP(\epsilon_r - 1) + 1 + (1 - 1/\epsilon_r)(CPMAX - CP) \qquad (4\text{-}4)\end{aligned}$$

Wheeler's [6] expression for ϵ_{eff} consists of two equations. The first is an interpolation to obtain the effective "filling fraction," and the second to obtain the effective dielectric constant. These equations are

$$q = q'' + \frac{q' - q''}{\epsilon_r} \qquad (4\text{-}5)$$

and

$$\epsilon_{eff} = q(\epsilon_r - 1) + 1 \qquad (4\text{-}6)$$

Substitution of q from (4-5) into (4-6) gives

$$\epsilon_{eff} = q''(\epsilon_r - 1) + 1 + (1 - 1/\epsilon_r)(q' - q'') \qquad (4\text{-}7)$$

In Wheeler's notation, q' is analogous to *CPMAX* and q'' is analogous to *CP*. With this notational change, (4-7) is exactly equivalent to (4-4), showing Wheeler's solution to be the first two terms of the binomial expansion of the solution derived in this paper for the case $R = 0.0$.

Wheeler derived his equations using approximate conformal mapping rather than exact conformal mapping as is done in this dissertation. Numerical results, using his derived formulation, are accurate to slightly over 1 percent of impedance, Z, compared to (3-23). When (4-7) is evaluated, however, with the exact conformal transformations derived in the appendixes, the result is within a few hundredths of a percent of the result from (3-23).

Both Dukes' and Wheeler's solutions are for the special case $R = 0.0$, balanced microstrip, whereas the solution derived in this paper is for generalized microstrip. The solutions are conceptually analogous with respect to the series expansion but are by no means equivalent.

Numerical Comparison

In addition to the analytical comparison with previous static solutions, it is instructive to compare numerical results. A complete comparison for the general case of microstrip is not possible because no solution, other than that derived here, exists. The case $R = 0.0$, balanced microstrip, and its image counterpart, unbalanced microstrip, with $R = 1.0$, can be compared since a number of solutions for these cases exist. Actually, the underlying concepts of both Dukes' and Wheeler's analyses can be directly applied to the general case using the general mapping equations derived in this dissertation Such a study has not, however, been formally done.

Three specific methods will be compared They are 1) Dukes' formulation [3] 2) Wheeler's approximate conformal formulation [7] and 3) the dielectric Green's function formulation of Weiss *et al.* [10]

Dukes' formulation is included because it shows the relative importance of the series component of dielectric Recall that Dukes omits the series dielectric component Comparison to Wheeler's approximate conformal method illustrates the difference between approximate and exact conformal methods The dielectric Green's function formulation is of interest for several reasons. It is based on sound analytic principles, it includes refraction effects and finally, the computer program is available

In the comparison with Dukes and Wheeler, which are both conformal, differences are due to approximations made by the analysts while differences from the dielectric Green's function result from numerical and discretization errors in computation and, to a lesser extent, from omission of refraction in the exact conformal mapping solution of this paper The dielectric Green's function method is the most important of the three since it includes refraction. A modified form of this method, which substantially removes numerical and discretization errors, will be used later to estimate the error in the conformal solution resulting from omission of refraction.

Figure 4-1 is a direct comparison of Z, computed using (3.23) of this paper with Dukes' formulation as given by eq. (4-3). Figure 4-2 compares eq. (9) of Wheeler [7], and Fig. 4-3 the computer program MSTRIP2, of Weiss *et al.*, [10]. Both Wheeler [7] and MSTRIP2 differ, at most, by slightly more than 1 percent. The maximum difference (for generalized microstrip) from Dukes [3] is slightly less than 0.75 percent.

Identification of the difference from Weiss *et al.* [10] as numerical or discretization errors, rather than errors due to omission of refraction, is done by comparison at $\epsilon_r = 1.0$ where the conformal solution is exact.

Estimate of Refraction Error

The total difference between the presence and absence of dielectric in the external region is small, so the effect on impedance of refraction at the dielectric boundaries must be much smaller yet. To quantify the error introduced by omission of refraction, a second solution, of equivalent precision to the conformal solution but including refraction, is needed. The difference in the results of these solutions would then approximate the refraction error in the conformal solution.

Basis for Refraction Error Estimate

The dielectric Green's function, as implemented in MSTRIP2, includes refraction but numerical and discretization errors are so large as to obscure its effect. To obtain a result of sufficient accuracy to permit comparison with the conformal solution of this paper in the range $\epsilon_r = 1$ to $\epsilon_r \to \infty$, the dielectric Green's function can be asymptotically corrected. Neither the conformal solution

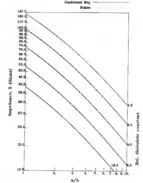


Fig. 4-1 Microstrip impedance computed by the method of Dukes, eq. (4-3), and by eq (3-23) of this dissertation versus conductor width and relative dielectric constant.

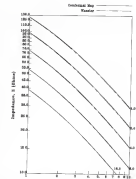


Fig. 4-2 Microstrip impedance computed by the approximate method of Wheeler, eq. (9) of Ref. 7, and eq. (3-23) of this dissertation versus conductor width and rel. diel. const.

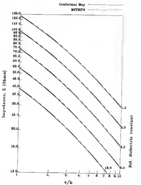


Fig 4-3 Microstrip impedance computed by the method of MSTRIP2 and by eq. (3-23) of this dissertation versus conductor width and relative dielectric constant

of this paper nor MSTRIP2 impose a limitation on the upper value of dielectric constant, so any very large value of ϵ_r within the floating point range of the computer can be used.

The solution of this dissertation provides the exact values of Z_0 and a/h at $\epsilon_r = 1$. The solution is also exact for very large ϵ_r. At $\epsilon_r = 10^{10}$ the error is on the order of 10^{-10} which certainly can be considered negligible.

Procedure for Calibrating MSTRIP2

The following exact parameters from the conformal solution will be used

I. @ $\epsilon_r = 1$, Z_0 (exact), a/h (exact)

II. @ $\epsilon_r = 10^{10}$, $Z \approx$ (exact), a/h (exact)

1. With inputs of Z_0 and with $\epsilon_r = 1$, find a/h with the conformal procedure

   NOTE. To obtain values of a/h suitable for precise comparison with MSTRIP2, the value of free-space impedance used in the conformal procedure, nominally 376.7 Ω, was set to correspond to the velocity of light used in MSTRIP2, 2.99792458×10^8 m/s

2. With inputs of $\epsilon_r = 1$, and $w/h = a/h$ to MSTRIP2, find Z. Let this be Z_{0w}

3. With inputs of Z_0 and $\epsilon_r = 10^{10}$, find Z with the conformal procedure

4. With inputs of $\epsilon_r = 10^{10}$ and $w/h = a/h$ to MSTRIP2, find Z. Let this be Z_w

5. Form the ratios $r' = Z_0/Z_{0w}$, and $r'' = Z/Z_w$ (4-8)

6. Find r for intermediate values of ϵ_r by interpolation in $1/\epsilon_r$ using

$$r = r'' + \frac{r' - r''}{\epsilon_r} \tag{4-9}$$

7. With the value of a/h used above, and any value of $1 < \epsilon_r < 10^{10}$ as input to MSTRIP2, compute the value of Z_w. Adjust the value of Z_w using

$$Z_w(\text{adjusted}) = rZ_w \tag{4-10}$$

8. With the value of Z_0 used above, and the value of $1 < \epsilon_r < 10^{10}$ used in the preceding step as input to the conformal procedure, compute the value of $Z(\epsilon_r)$.

9. Compute the percent difference, δ, as

$$\delta = \left[\frac{Z(\epsilon_r)}{Z_w(\text{ adjusted})} - 1\right] \times 100 \tag{4-11}$$

Refraction Error Estimate

The value of Z_w(adjusted) from MSTRIP2 is always slightly lower than the value of Z computed by the conformal method. The maximum difference of $\delta = 0.0874$ percent occurs at or near $Z_0 = 111.0\ \Omega$ ($a/h = 1.3183874$) and $\epsilon_r = 3.6$. This difference is tentatively attributed to the omission of refraction effects in the conformal computation of Z.

The correction technique gives consistent results over the range $10.0 \le Z_0 \le 500.0\ \ \Omega$ using MSTRIP2 with $m = 20$. MSTRIP2 does not support computation of generalized microstrip but the results obtained for a single conducting strip over an infinite ground plane $R = 1.0$, are thought to be reasonably representative.

Spurious Modes

Edwards [14] gives formulas for the cutoff frequencies for the TM and transverse resonance modes. Edwards cites these formulas from Vendelin (reference 37 of Edwards book) who, in turn, attributes Collin. The formula for TM cutoff is

$$f_{TM1} = \frac{c \arctan(\epsilon_r)}{\pi t \sqrt{2(\epsilon_r - 1)}} \tag{4-12}$$

where c is the free-space velocity of light and $t = 2h$, the substrate thickness. Transverse resonance cutoff is approximated by

$$f_{CT} = \frac{c}{(2w + 0.8t)\sqrt{\epsilon_r}} \tag{4-13}$$

with $w = 2a$.

Relationship to the Wave Equation

Assuming that the conditions implied by (4-12) and (4-13) are met, the relative error with respect to the wave solution can be estimated, based on rather simple assumptions. The primary source of error in the static solution with respect to the wave solution is the inhomogeneous dielectric which gives rise to a disparate propagation velocity for the field fractions terminating on the inner and outer faces of the conductors. If the series component of the external field is neglected, propagation is at the velocity of light in free-space. Conversely, the field terminating on the inner faces is entirely in dielectric with a velocity of propagation relative to free-space vp, assuming that each field exists independent of the other. This relative velocity can be written

$$vp = \frac{v_d}{c} = \frac{1}{\sqrt{\epsilon_r}} \tag{4-14}$$

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where v_g is the absolute propagation velocity in the dielectric. Using (4-14) the relative velocity differential between the inner and outer fields can be defined as

$$\delta(\epsilon_r) = (c - v_g)/c = 1 - vp \tag{4-15}$$

The relative velocity differential is seen to be zero for $\epsilon_r = 1$ and approaches a maximum of 1 as ϵ_r is increased to a very large value. The internal and external capacitances are given by eqs. (3-17) and (3-21), respectively, and their ratio is a measure of the field fractions terminating on the inner and outer faces of the conductors. Define

$$f(\epsilon_r) = C_e/C_i \tag{4-16}$$

From eq. (3-22) the external capacitance (C_e) has a finite maximum value, which it reaches at the limit for large ϵ_r, while the internal capacitance increases without limit as ϵ_r increases without limit. As $\epsilon_r \to \infty$, $f(\epsilon_r) \to 0$, meaning that the external field is negligible with respect to the internal field for very large ϵ_r.

At the limit for low impedance the conductor widths increase without limit, ultimately becoming an infinite radial transmission line with the field entirely on its inner faces. Therefore, at low impedance, the velocity differential is meaningless since there is no field on the outer faces. If it is assumed that the error in the static approximation is proportional to $\delta(\epsilon_r)$ and to the ratio of the external to internal capacitance $f(\epsilon_r)$, the estimate of error in the static approximation with respect to the wave solution can be written

$$\text{error} = c_1\delta(\epsilon_r)f(\epsilon_r) \tag{4-17}$$

where c_1 is an undetermined proportionality constant.

When $\epsilon_r = 1$, $h(\epsilon_r) = 0$ and the error is zero by virtue of the homogeneous dielectric. When $\epsilon_r \rightarrow \infty$, the external field is negligible and the error is zero by virtue of homogeneous dielectric. At very low impedance the external field is zero and the error is zero independent of dielectric constant. The accurate range of calculation, without resorting to asymptotic approximations, is approximately 13.0 to 840.0 Ohms. The error for this range, relative to c_1, is

$$rel\ error = (1 - \frac{1}{\sqrt{\epsilon_r}}) \frac{\left(\frac{1-A'}{1-(1-\frac{1}{\epsilon_r})\frac{CPMAX-CP}{1-A'}} \right)}{CP(\epsilon_r - 1) + A'} \tag{4-18}$$

Calculations made with (4-18) give the error relative to the unknown constant c_1. To find the absolute error it is necessary to evaluate the proportionality constant which requires a wave solution not currently available. Evaluation of (4-18) for the relative error is of some interest since it indicates the relative magnitude of error as a function of $Z0$, R and ϵ_r. Also since c_1 is a particular single constant, the location of maximum of the relative error will also be that of maximum of the absolute error. Numerical results for these calculations will be presented in the next chapter.

CHAPTER 5
GENERALIZED MICROSTRIP CALCULATIONS AND RESULTS

Demonstration Programs

In the preceding Chapters and the Appendixes A and B, about two hundred equations have been derived. The thirty six most frequently employed are collected, by category, in Table 5-1 for reference. Two demonstration programs used in preparation of this dissertation, entitled MIST1 and MIST2, together with their supporting subroutines and function subroutines, are listed in Appendix C. These programs are written in double precision Fortran 77 and are designed for use with "G" floating point format. They should also function in "D" floating point format but possibly over a slightly reduced range of impedance. The programs and subprograms are extensively commented and are self documenting.

Program MIST1

Program MIST1.FOR accepts the free-space impedance Z_0, the parameter R and the relative dielectric constant ϵ_r as inputs and returns the corresponding values of a/h, b/h, Z, vp and the transmission line capacitance per meter in PF for generalized microstrip on a dielectric sheet. The calculation flow is straightforward and is readily followed from the program.

TABLE 5-1
SUMMARY OF FORMULAS

Capacitance and Impedance

$$c_0 = \epsilon_0 \frac{K'}{K} \qquad \text{(A-11)}$$

$$Z_0 = \eta \frac{K}{K'} \qquad \text{(A-12)}$$

with $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7\ \Omega$

Formulas for Generalized Microstrip

Constants

$$C2 - C1 = \frac{R(2E' - K' - k^2K')}{R^2E' - K'} \qquad \text{(A-48)}$$

$$C1 \times C2 = \frac{R^2k^2K' - E'}{R^2E' - K'} \qquad \text{(A-49)}$$

Transform Functions

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) + \frac{\mathrm{cn}\,w\,\mathrm{dn}\,w}{\mathrm{sn}\,w + 1/Rk}\right] \qquad \text{(A-55)}$$

$$X = x/h = \frac{u}{K} + \frac{2K'}{\pi}\left[E(u,k) - \frac{E}{K}u + \frac{k^2scds_1^2}{\Delta} + \frac{R^2k^2\{RkA + (AC - BD)/\Delta\}}{(RkC + \Delta)^2 + (RkD)^2}\right] \qquad \text{(A-64)}$$

$$Y = y/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}v - E(v,k') + \frac{s_1c_1d_1d^2}{\Delta} - \frac{R^2k^2\{RkB + (AD - BC)/\Delta\}}{(RkC + \Delta)^2 + (RkD)^2}\right] \qquad \text{(A-65)}$$

$A = cdd_1(c_1^2 - k^2s^2s_1^2)$, $B = ss_1c_1(k^2c^2 + d^2d_1^2)$, $C = sd_1$, $D = cds_1c_1$, $\Delta = 1 - d^2s_1^2$,

with $s = \mathrm{sn}(u,k)$, $c = \mathrm{cn}(u,k)$ $d = \mathrm{dn}(u,k)$ $s_1 = \mathrm{sn}(v,k')$, $c_1 = \mathrm{cn}(v,k')$ and $d_1 = \mathrm{dn}(v,k')$

TABLE 5-1 (Continued)

Shape Parameters

$$T_1 = \frac{1}{k}\sqrt{\frac{C1^2 - k^2}{1 - C1^2}}, \tag{A-96}$$

$$a' = F(T_1, k'), \tag{A-97}$$

with $[K'/2 \le a' \le K']$.

$$a/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}F(T_1,k') - E(T_1,k') + T_1 k(1 - C1^2)\left(\frac{1}{C1} - \frac{R}{1 + RC1}\right)\right] \tag{A-98}$$

$$T_2 = \frac{1}{k}\sqrt{\frac{C2^2 - k^2}{1 - C2^2}}, \tag{A-99}$$

$$b' = F(T_2, k'), \tag{A-100}$$

with $[K'/2 \le b' \le K']$, and $[b' \ge a']$

$$b/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}F(T_2,k') - E(T_2,k') + T_2 k(1 - C2^2)\left(\frac{1}{C2} + \frac{R}{1 - RC2}\right)\right] \tag{A-101}$$

Parameter u_0

$$u_0 = F(T_3, k) \tag{A-110}$$

$$T_3 = \sqrt{\frac{R^2}{1 - R^2}} \tag{A-109}$$

Formulas for the Special Case $R = 0$

Transform Functions

$$x(w)\big|_{R=0} = \frac{2kK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w)\right] \tag{A-31}$$

$$X = x/h\big|_{R=0} = \frac{u}{K} + \frac{2K'}{\pi}\left[E(u,k) - \frac{E}{K}u + \frac{k^2 s c s_1^2}{\Delta}\right] \tag{A-62}$$

$$Y = y/h\big|_{R=0} = \frac{2K'}{\pi}\left[\frac{E'}{K'}v - E(v,k') + \frac{s_1 c_1 d_1 d^2}{\Delta}\right] \tag{A-63}$$

TABLE 5.1 (Continued)

Inverse Transform Functions

To find v = v(X,u,k)

$$v(X,u,k) = F(T_u, k'), \tag{A-71}$$

$$T_u = \frac{1}{k}\sqrt{\frac{P}{scd - Ps^2}}, \tag{A-70}$$

$$P = \frac{\pi}{2K}\left[X - \frac{u}{K}\right] - E(T_u,k) + \frac{E}{K}u, \tag{A-72}$$

$$T_u = \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u} \tag{A-73}$$

To find u = u(Y,v,k)

$$u(Y,v,k') = F(T_v,k) \tag{A-75}$$

$$T_v = \sqrt{\frac{s_1c_1d_1 - Qc_1^2}{Qd_1^2 - k^2s_1c_1d_1}}, \tag{A-74}$$

$$Q = \frac{\pi}{2K'}Y + E(T_v,k') - \frac{E'}{K'}v, \tag{A-76}$$

$$T_v = \frac{\mathrm{sn}\,v}{\mathrm{cn}\,v} \tag{A-77}$$

Shape Parameters

$$C0^2 = \frac{E'}{K'} \tag{A-102}$$

$$T_0 = \frac{1}{k}\sqrt{\frac{C0^2 - k^2}{1 - C0^2}}, \tag{A-103}$$

$$a\,|_{X=0} = b = F(T_0,k'), \tag{A-104}$$

TABLE 5-1 (Continued)

$$a/h\,|_{b=a} = b/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}F(T_0,k') - E(T_0,k') + \frac{T_0k(1-C0^2)}{C0}\right] \qquad \text{(A-105)}$$

Dielectric Polarization Capacitance

$$c_p = \epsilon_0 \frac{2K(p)}{K'(p)}(\epsilon_r - 1) \qquad \text{(B-11)}$$

$$p = \tanh\left(\frac{\pi a}{4h}\right)\tanh\left(\frac{\pi b}{4h}\right) \qquad \text{(B-5)}$$

$$p' = \sqrt{\operatorname{sech}^2 A + \operatorname{sech}^2 B - \operatorname{sech}^2 A \operatorname{sech}^2 B} \qquad \text{(B-6)}$$

where $A = \pi a/4h$, and $B = \pi b/4h$

Effective Dielectric Constant

$$\epsilon_{eff} = CP(\epsilon_r - 1) + A' + \frac{1 - A'}{1 - (1 - \frac{1}{\epsilon_r})\frac{CPMAX - CP}{1 - A'}} \qquad \text{(3-23)}$$

Approximation

$$\epsilon_{eff} = CP(\epsilon_r - 1) + A' + \frac{1 - A}{1 - 0.536(1 - \frac{1}{\epsilon_r})\frac{CP - A}{1 - A'}} \qquad \text{(3-28)}$$

Characteristic Impedance on a Dielectric Sheet

$$Z = \frac{Z_0}{\sqrt{\epsilon_{eff}}} \qquad \text{(3-27)}$$

Figure 5.1 is a graph of generalized microstrip for values obtained for *Z* from 10.0 to 200.0 Ohms and *R* from 0.0 to 0.99 with $\epsilon_r = 9.6$, 99.5 per cent alumina. Any number of such graphs for general values of *R*, and various values of dielectric constant, can be computed with high accuracy.

Tables 5.2 and 5-3 are estimates of the maximum relative error of the static solution with respect to a wave solution. The values of maximum relative error, given in Table 5-2, are relative to an undetermined constant. Table 5-3 indicates that the maximum difference between the results of this dissertation and a wave solution should occur near a relative dielectric constant of 2.3. This information is useful as a beginning point for future comparison to a wave solution and also to avoid, wherever possible, the region of potential error.

Program MIST1 computes the static parameters of generalized microstrip in their most accurate form, using eq. (3-23), including integration of the area to obtain *CPMAX*. In addition, the program simultaneously computes impedance using the approximate formulation, eq. (3-28), and the percentage difference between the resulting impedance values. The peak difference is shown to be less than 0.25 per cent, more than adequate for all practical calculations. The computationally intensive operations for calculation of dielectric area are not required when using eq. (3-28). This can result in a significant simplification and speed increase for generalized microstrip design programs.

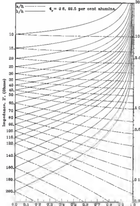

Fig. 5-1 Conductor widths versus impedance and R for generalized microstrip on alumina

TABLE 5-2
MAXIMUM RELATIVE ERROR VERSUS ZD AND *R*

ZD						
840 000	0.145	0.145	0.145	0.145	0.145	0.147
420 000	0.134	0.136	0.137	0.139	0.140	0.145
210 000	0.096	0.103	0.110	0.117	0.125	0.134
105 000	0.056	0.063	0.071	0.080	0.089	0.096
52 500	0.031	0.036	0.042	0.048	0.053	0.056
26 250	0.017	0.021	0.025	0.027	0.029	0.031
13 125	0.009	0.012	0.014	0.015	0.016	0.017
R	0.0	0.2	0.4	0.6	0.8	1.0

TABLE 5-3
e_r FOR MAXIMUM RELATIVE ERROR VERSUS ZO AND R

ZO						
840 000	2 227	2.227	2.227	2.226	2 226	2 239
420 000	2.220	2 212	2.207	2.203	2 203	2 227
210 000	2.251	2 227	2 212	2.207	2 216	2 220
105 000	2 285	2 263	2 260	2 269	2.275	2 251
52 500	2.500	2 292	2 309	2 321	2.312	2 285
26 250	2 304	2 312	2 339	2 339	2.320	2.300
13 125	2 303	2 328	2 348	2 334	2.317	2.304
R	0 0	0 2	0 4	0 6	0 8	1 0

Program MIST2

Program MIST2 FOR accepts the desired impedance (Z) of the microstrip on a dielectric sheet and the relative dielectric constant ϵ_r, as inputs and returns the conductor width *a/h*, vp and the transmission line capacitance per meter in pico Farads, selectably for either unbalanced or balanced microstrip Since the input is the impedance on a dielectric sheet *Z* rather than Z_0 the program iterates to the appropriate value of Z_0, a method that is required for many design calculations in generalized microstrip As was the case for MIST1, this program implements the most accurate form of the derived equations Integration of dielectric area is by use of the inverse transformation functions, eqs (A 70), (A-71), (A-72) and (A 73), for finding the dielectric boundary These equations do not require iteration and are sufficiently fast for direct inclusion in programs which iterate on other parameters.

Program Listings

The Fortran source listings of the programs MIST1 and MIST2 in Appendix C have been used extensively by the author and appear to be quite robust The author will be most appreciative of comments, suggestions and, in particular, description of any errors that might be found. The first listings are of the two main programs MIST1 and MIST2, followed by the subroutines in alphabetical order

Programs Not Listed

Three of the required subroutines, DEL2, DQG32 and DZERO are from the open literature. These programs are copyrighted, and hence are referenced, but not listed. Two of the unlisted programs, DEL2 and DZERO, require modification. The required changes are listed in Appendix C.

CHAPTER 6
CONCLUSIONS

Accuracy

Static Solution

The microstrip equations derived in this dissertation, and implemented in the computer programs, are thought to be at, or near, the accuracy limit for static techniques. The absolute static impedance error is estimated to be on the order of 0.001 Z and almost entirely attributable to the omission of refraction effects. The derived equations are exact for free space and at the limit for high dielectric constant. In between, the effect of the series component follows bound charge which is proportional to $(1 - 1/\epsilon_r)$. This is a derived relationship based on the assumption that the series field component is uniformly distributed above the parallel component to eliminate refraction. This assumption properly includes all of the effects of the series dielectric, with the exception of variation of the static field distribution in the series region for intermediate values of ϵ_r. The parallel component of capacitance is exactly characterized by the equations.

Accuracy with Respect to a Wave Solution

A further implication of high accuracy can be seen from the ratio of external to internal capacitance. For alumina, over 92 per cent of the field is entirely in dielectric for unbalanced microstrip and over 95 per cent for balanced which strongly implies that propagation is accurately characterized by the static equations for alumina. For lower dielectric constant a greater proportion of the field is in the mixed dielectric (series) region but the disparity in velocity between the internal and external fields is also smaller. These effects are analyzed in Chapter 5 to obtain a relative estimate of the difference between the static solution of this dissertation and a wave solution. This analysis indicates that the maximum difference from a wave solution should be in the neighborhood of $\epsilon_r = 2.3$. Unfortunately, the analysis cannot provide absolute differences but these are thought to be small.

Future Work

In years past, the author has done extensive experimental work on generalized microstrip, particularly on applications relating to the design of balun transformers but using less accurate design methods. There is a need for updated experimental verification of the accuracy estimates for the theory of this dissertation, particularly with respect to wave propagation. There is also a need for development of more comprehensive and efficient design software for generalized microstrip.

For some applications, capacitive coupling (DC blockage) is required for use of the balun transformer. An investigation of coupling methods for tapered line balun transformers could lead to improvement in performance for a variety of applications.

The direct application of balun transformers as feeds for spiral antennas and for doubly balanced mixers have been the principal applications for generalized microstrip. A potential area for further application is to the design of components in balanced microstrip using baluns for input and output. Components such as branch line couplers can be designed in balanced microstrip, possibly with considerable benefit. As an example, using a 0.010 inch alumina substrate, the conductor width, for 100.0 Ω unbalanced microstrip, is 0.00141 inches whereas for balanced microstrip it is 0.00496 inches, or nearly 5 times as wide. At higher impedance the difference is even more pronounced. At 180.0 Ω unbalanced microstrip gives a conductor width of 0.000065 inches, virtually unrealizable, whereas balanced microstrip, for the same impedance, has a width of 0.00104 inches, over 15 times wider than its unbalanced counterpart and nearly as wide as unbalanced microstrip at 100.0 Ω. This raises the possibility of fabricating balanced branch line couplers with more sections, and thus more bandwidth, than could be achieved in unbalanced microstrip. An investigation of design applications for balanced microstrip could lead to significant developments.

Other Strip Configurations

The careful reader has probably noted that the basic theory, used here for generalized microstrip, can be applied to other problems involving conducting strips on a dielectric sheet. As an example, coplanar configurations are of significant interest and should be readily analyzable by the methods of this dissertation.

APPENDIX A
CONFORMAL TRANSFORMATION OF GENERALIZED MICROSTRIP

The exact conformal transformations of generalized microstrip result in equations involving specialized integrals and functions. For those unfamiliar with conformal mapping, the first section is a brief outline of conformal transformations, the Schwarz-Christoffel transformation, and the special integrals and functions that arise in the derivation of the transformations

Conformal Transformations

Consider the relationship between the z and w-planes:

$$z = F(w)$$

where $z = x + iy$ and $w = u + iv$. If F is an analytic function i.e., satisfies the Cauchy-Riemann conditions, $\partial x/\partial u = \partial y/\partial v$, $\partial x/\partial v = -\partial y/\partial u$, then the transformation is conformal Angles are preserved by the transformation and squares in the w-plane map into curvilinear squares in the z-plane To obtain this transformation the function F can be chosen at random, provided it is analytic, or it may be found by the Schwarz-Christoffel method which guarantees it to be analytic Separation of $z = F(w)$ into real and imaginary parts gives the two equations,

$$x = \xi_1(u,v) \text{ and } y = \xi_2(u,v)$$

These equations can be used in plotting the two families of curves u = constant and v = constant, which form an orthogonal map in the z-plane In some cases the inverse transformation

$$u = \zeta_1(x,y) \text{ and } v = \zeta_2(x,y),$$

can be found, permitting bi-directional mapping of points between the two planes. In those cases where the inverse transformation cannot be found, bi-directional mapping can usually be done iteratively.

Conformal transformations have been widely used in the study of two-dimensional field problems, particularly in electrostatics. Both stripline and microstrip are generally well suited by their inherent geometry for analysis by the Schwarz-Christoffel method. This is not always the case, however, and some important microstrip configurations cannot be directly transformed by the method.

The difficulties arising in the integration of the Schwarz-Christoffel polynomial for complicated polygons are well known. The number of significant geometries that have been solved in closed form since around 1870, when Schwarz and Christoffel published, probably number no more than a few hundred and possibly much less. For those cases where closed form solutions cannot be obtained, numerical integration or series expansion of the transformation can be used. These methods, however, lose many of the advantages of the closed analytic solution and there are usually better numerical methods available than numerical integration of the Schwarz-Christoffel polynomial. This difficulty in obtaining a closed form analytic solution is one of the primary drivers of research into modern numerical methods. Urgent problems, usually, cannot wait for an analytic solution. The few geometries that have been formulated in closed form are a valuable resource. They provide not only design equations but also measurement standards for numerical methods

Restrictions

Conformal mapping, including the Schwarz-Christoffel transformation, is restricted to two dimensions. The potential solutions obtained are those for the two-dimensional Laplace equation and are exact only for purely transverse fields and homogeneous dielectric. For purely transverse propagating waves i.e., pure TEM waves, the two-dimensional Laplace solution is also a solution for the one-dimensional wave equation. Application of the Laplace solution to waves for the inhomogeneous dielectric case requires the quasi-TEM assumption which principally involves assuming that there are negligible higher order TE or TM modes and negligible losses or, stated more generally, that there be negligible longitudinal field from any source. Refraction at the dielectric boundaries cannot be treated by conformal mapping which restricts solutions to homogeneous dielectric unless refraction effects are small. In addition to these restrictions, there is a practical limitation on the number of right angle vertices in the polygon. This is due to the mathematical difficulty of integrating the resulting polynomial equation. To minimize the number of vertices, conductors are usually assumed to be thin. The following summarizes the usual assumptions

1. The longitudinal electric field is negligible.
2. The conductors are immersed in a homogeneous dielectric.
3. Conductor and dielectric losses are negligible

4. Conductor thickness is negligible

The Schwarz-Christoffel Transformation

The Schwarz-Christoffel transformation maps the interior of an arbitrary closed polygon onto a half-plane. There are certain formal restrictions on the nature of the polygon but within those restrictions the transformation can always be obtained in polynomial form. Once the polynomial function is obtained, the transform function can be discovered by integration provided that it can be integrated. The procedure for obtaining the Schwarz-Christoffel derivative can be stated in the following rules:

1. Traverse the polygon perimeter, maintaining the interior on the left.
2. Note the internal angles, α_n, at each of the n vertices.
3. The exponents, μ_n, of the resulting polynomial function are given by

$$\mu_n = \alpha_n/\pi - 1$$

4. The sum of the external angles, after traversing the entire periphery, must be 2π or, equivalently, the sum of the exponents must be -2
5. If the polygon is in the z-plane and the t-plane is the transform half-plane,

then $$\frac{dz}{dt} = D(t - t_1)^{\mu_1}(t - t_2)^{\mu_2} \cdots (t - t_n)^{\mu_n}$$

where D is a constant, possibly complex.

Elliptic Integrals

The nature of the transform functions arising from integration of the Schwarzian derivative depend on the angles of the polygon's vertices. Angles $\alpha_n = \pm\pi/2$, right angles, result in exponents $\mu_n = \mp 1/2$, the square root of the branch point. These branch points are of particular interest since they may result in elliptic integrals. There are exceptions but in general the following rules will correctly predict the nature of the transform function.

1. Up to two right angles result in elementary functions.
2. Three or four right angles result in elliptic integrals.
3. Five or more right angles result in hyperelliptic integrals.

The transformations for generalized microstrip fall in the second category, involving elliptic integrals and elliptic functions. For those unfamiliar with these integrals and functions the following descriptions, notation and definitions will assist in following the derivations.

There are three types of elliptic integrals designated as first, second and third kind. Additionally each can be incomplete or complete. Notationally these are usually written as shown below.

Elliptic Integral of the First Kind

$F(\varphi,k)$ - incomplete elliptic integral of the first kind, with amplitude φ, and modulus k

$$F(\varphi,k) \triangleq \int_0^t \frac{d\xi}{\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}} = \int_0^\varphi \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \qquad \text{(A-1)}$$

The first form of the integral is known as the Jacobian form and the second is the Legendre form. The Legendre form is obtained from the Jacobian by the substitution $t = \sin\varphi$. Elliptic integrals of the first, second and third kind are said to be complete when the upper limit of the Legendre form $\varphi = \pi/2$.

$F(\pi/2,k) = K(k)$ – complete elliptic integral of the first kind, with amplitude $\pi/2$ and modulus k, usually written K.

Elliptic Integral of the Second Kind

$E(\varphi,k)$ – incomplete elliptic integral of the second kind, with amplitude φ and modulus k

$$E(\varphi,k) \triangleq \int_0^t \frac{\sqrt{1-k^2\xi^2}}{\sqrt{1-\xi^2}}\,d\xi = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{A 2}$$

$E(\pi/2,k) = E(k)$ – complete elliptic integral of the second kind, with amplitude $\pi/2$, and modulus k, usually written E

Elliptic Integral of the Third Kind

$\Pi(\varphi,\alpha^2,k)$ – incomplete elliptic integral of the Third kind, with amplitude φ, parameter α^2 and modulus k

$$\Pi(\varphi,\alpha^2,k) \triangleq \int_0^t \frac{d\xi}{(1-\alpha^2\xi^2)\sqrt{1-\xi^2}\,\sqrt{1-k^2\xi^2}} = \int_0^\varphi \frac{d\theta}{(1-\alpha^2\sin^2\xi)\sqrt{1-k^2\sin^2\xi}} \tag{A-3}$$

$\Pi(\pi/2,\alpha^2,k)$ – complete elliptic integral of the third kind, with amplitude $\pi/2$, parameter α^2, and modulus k

The Modulus

The modulus k can take any real, imaginary or complex value. In engineering applications, however, it is usual to take $0 \le k \le 1$. The complementary modulus k' is related to the modulus by

$$k' = \sqrt{1-k^2}$$

Complete elliptic integrals of the first and second kind with modulus k' are usually written K' and E'

The Amplitude

The upper integration limit is called the amplitude t or φ, of elliptic integrals. Like the modulus, it can be real imaginary or complex but are usually taken $0 < t \leq 1$ and $0 < \varphi \leq \pi/2$. Notationally, am u = $\mathrm{am}(u,k) = \varphi$, [$u$ real] and $\mathrm{am}^{-1}(\varphi,k) = \mathrm{am}^{-1}(\sin^{-1}t,k) = F(\varphi,k)$

Elliptic Functions

The inverse of the elliptic integral of the first kind was investigated by Jacobi and Abel. Jacobi called this inverse function the "sinus amplitudinus," now more commonly called the sine amplitude u and written $\mathrm{sn}(u,k)$ or sn u. The inverse sine, or arcsine, is defined by the equation

$$\arcsin t = \int_0^t \frac{d\xi}{\sqrt{1-\xi^2}}, \quad [0 < t \leq 1]$$

By analogy with the circular functions the inverse sine amplitude is defined by the equation

$$\mathrm{arcsn}\,(t,k) = \int_0^t \frac{d\xi}{\sqrt{1-\xi^2}\,\sqrt{1-k^2\xi^2}}, \quad [0 < t \leq 1]$$

The inverse relationships are

$$u = \mathrm{sn}^{-1}(t,k) = F(\sin^{-1}t,k) = F(\varphi,k)$$

and the corresponding direct relationships are defined by

$$t = \mathrm{sn}(u,k) = \sin\varphi, \text{ and } \varphi = \mathrm{am}(u,k) \tag{A-4}$$

Like the trigonometric functions, the sine amplitude has a real period and, like the hyperbolic functions, an imaginary period. The real period is $4K$ and the imaginary period is $2iK'$. Two additional elliptic functions can be defined in terms of the sine amplitude. The "cosinus amplitudinus" or cosine amplitude, $\mathrm{cn}\, u = \cos\varphi = (1 - \mathrm{sn}^2 u)^{1/2}$, and the "deltas amplitudinus," or delta amplitude, $\mathrm{dn}\, u = (1 - k^2\sin^2\varphi)^{1/2}$. These have periods $(4K, 2K + 2iK')$ and $(2K, 4iK')$, respectively. Additional elliptic functions are defined in terms of the reciprocals and quotients of these three basic functions, giving twelve elliptic functions in all. In the derivations which follow, only the three basic elliptic functions and the tangent amplitude, $\mathrm{tn}\, u = \mathrm{tn}(u,k) = \mathrm{sn}\, u/\mathrm{cn}\, u$, will be used.

The elliptic functions are usually written $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$ and $\mathrm{tn}\, u$ rather than $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$, and $\mathrm{tn}(u,k)$. If the argument is v, they become $\mathrm{sn}\, v$, $\mathrm{cn}\, v$, $\mathrm{dn}\, v$, and $\mathrm{tn}\, v$ instead of $\mathrm{sn}(v,k')$, $\mathrm{cn}(v,k')$, $\mathrm{dn}(v,k')$, and $\mathrm{tn}(v,k')$. Specifying the modulus in these cases is unnecessary since k always goes with u and k' with v.

Notation

The notation used for elliptic integrals and functions is notoriously chaotic and unstandardized. For this reason a single reference was selected, "Byrd and Friedman [15]." At appropriate points in the following text the notation "B F *nnn.nn*" appears following an equation, meaning that the source of the equation or identity is Byrd and Friedman's equation *nnn.nn*. Departures from their notation will be noted where they occur. An alternative reference, which may be more generally available than Byrd and Friedman, is the "Table of Integrals, Series and Products" by Gradshteyn and Ryzhik [16]. Their treatment of elliptic integrals and functions is quite thorough, though not as complete as Byrd and Friedman.

The Schwarz-Christoffel Transformations for Generalized Microstrip

The z-plane to w-plane transformation is obtained through the use of an intermediate t-plane transformation with branch points common to both the z and w-planes. This transformation was obtained by Black and Higgins [2] using values at the branch points different from those used here. Their solution can be transformed to obtain the result derived here; it is useful and illustrative, however, to carry out the entire derivation independent of their prior solution.

The three transform planes are illustrated in Fig. A-1 and the transform points are tabulated in Table A-1. Note that the sum of the exponents is −2 for each of the transformations as required.

In the process of deriving the transforms of generalized microstrip, a number of definitions and identities relating to elliptic integrals and functions will be required. All of these can be found in the suggested references. For convenience, however, the applicable relationships have been collected in Table A-2. These formulas are numbered 1-nn and will be referred to by number as needed in the text.

The w to t plane Transformation

Using the internal angles of the polygon in the w-plane and traversing counter-clockwise from (2) to (7) and back to (2), in Fig. A-1, gives the Schwarzian differential

$$dw = D''\overset{(2)}{(t + 1/k)}^{-1/2}\,\overset{(4)}{(t + 1)}^{-1/2}\,\overset{(5)}{(t - 1)}^{-1/2}\,\overset{(7)}{(t - 1/k)}^{-1/2}\,dt$$

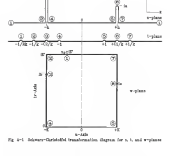

Fig A-1 Schwarz-Christoffel transformation diagram for z, t, and w-planes

TABLE A-1
Tabulation of transform points.

Point	①	②	③	④	⑤	⑥	⑦
z-plane	Inf.	-h	-h+ib	-h	+h	+h+ia	+h
t-plane	-1/Rk	-1/k	-C2/k	-1	+1	+C1/k	+1/k
Exponent	-2	-1/2	+1	-1/2	-1/2	+1	-1/2
w-plane	u_0+iK'	-K+iK'	-K+ib	-K	+K	+K+ia	+K+iK'
t-plane	-1/Rk	-1/k	-C2/k	-1	+1	+C1/k	+1/k
Exponent	0	-1/2	0	-1/2	-1/2	0	-1/2

TABLE A-2
NOTATION, DEFINITIONS AND IDENTITIES FOR ELLIPTIC INTEGRALS AND FUNCTIONS

Notation

$F(\pi/2,k) \equiv K \quad E(\pi/2,k) \equiv E \quad F(\pi/2,k') \equiv K' \quad E(\pi/2,k') \equiv E'$

$\mathrm{sn}(u,k) \equiv \mathrm{sn}\, u \equiv s,\ \mathrm{cn}(u,k) \equiv \mathrm{cn}\, u \equiv c,\ \mathrm{dn}(u,k) \equiv \mathrm{dn}\, u \equiv d.$

$\mathrm{sn}(v,k') \equiv \mathrm{sn}\, v \equiv s_1,\ \mathrm{cn}(v,k') \equiv \mathrm{cn}\, v \equiv c_1,\ \mathrm{dn}(v,k') \equiv \mathrm{dn}\, v \equiv d_1$

Identities, Formulas and Definitions

I-1 $\mathrm{am}\, u \equiv \mathrm{am}(u,k) = \varphi \quad (u \text{ real})$ (Direct)
I-2 $\mathrm{am}^{-1}(\varphi,k) \equiv \mathrm{am}^{-1}(\sin^{-1} t,k) = F(\varphi,k) \equiv F(\sin^{-1} t,k)$ (Inverse)
I-3 $t = \mathrm{sn}\, u \equiv \mathrm{sn}(u,k) = \sin\varphi$ (Direct)
I-4 $u = \mathrm{sn}^{-1}(t,k) = F(\varphi,k) \equiv F(\sin^{-1} t,k)$ (Inverse)
I-5 $\mathrm{tn}\, u = \mathrm{sn}\, u/\mathrm{cn}\, u$
I-6 $k^2 + k'^2 = 1$

$-1 \le \mathrm{sn}\, u \le 1,\ -1 \le \mathrm{cn}\, u \le 1,\ k' \le \mathrm{dn}\, u \le 1,\ -\infty < \mathrm{tn}\, u < \infty$

I-7 $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1,$ $\quad \mathrm{sn}^2 v + \mathrm{cn}^2 v = 1,$
I-8 $k^2\mathrm{sn}^2 u + \mathrm{dn}^2 u = 1,$ $\quad k'^2\mathrm{sn}^2 v + \mathrm{dn}^2 v = 1,$
I-9 $\mathrm{dn}^2 u - k^2\mathrm{cn}^2 u = k'^2,$ $\quad \mathrm{dn}^2 v - k'^2\mathrm{cn}^2 v = k^2,$
I-10 $k'^2\mathrm{sn}^2 u + \mathrm{cn}^2 u = \mathrm{dn}^2 u,$ $\quad k^2\mathrm{sn}^2 v + \mathrm{cn}^2 v = \mathrm{dn}^2 v$

I-11 $\mathrm{am}(-u) = -\mathrm{am}\, u,$ $\quad$ I-16 $\mathrm{am}\, K = \pi/2,$
I-12 $\mathrm{sn}(-u) = -\mathrm{sn}\, u,$ $\quad$ I-17 $\mathrm{sn}\, K = 1,$
I-13 $\mathrm{cn}(-u) = \mathrm{cn}\, u,$ $\quad$ I-18 $\mathrm{cn}\, K = 0,$
I-14 $\mathrm{dn}(-u) = \mathrm{dn}\, u$ $\quad$ I-19 $\mathrm{dn}\, K = k',$
I-15 $\mathrm{tn}(-u) = -\mathrm{tn}\, u$ $\quad$ I-20 $\mathrm{tn}\, K = \infty$

I-21 $\mathrm{sn}(u \pm iv) = (sd_1 \pm icds_1c_1)/\Delta,$ $\quad \Delta = 1 - d^2s_1^2$
I-22 $\mathrm{sn}(K + iK') = 1/k,$ $\quad \mathrm{sn}(-K + iK') = -1/k$
I-23 $\mathrm{cn}(u \pm iv) = (cc_1 \mp isds_1d_1)/\Delta,$
I-24 $\mathrm{cn}(K + iK') = -ik'/k,$ $\quad \mathrm{cn}(-K + iK') = -ik'/k$
I-25 $\mathrm{dn}(u \pm iv) = (dc_1d_1 \mp ik^2scs_1)/\Delta.$
I-26 $\mathrm{dn}(K + iK') = 0,$ $\quad \mathrm{dn}(-K + iK') = 0$

I-27 $E(u \pm iv) = E(u,k) + (k^2scds_1^2)/\Delta \pm i[v - E(v,k') + (s_1c_1d_1d^2)/\Delta],$
I-28 $E(K + iK') = E + i(K' - E'),$ $\quad E(-K + iK') = -E + i(K' + E')$
I-29 $E(-u,k) = -E(u,k)$ $\quad$ I-31 $E(K) = E,$
I-30 $EK' + E'K - KK' = \pi/2,$ $\quad$ I-32 $E(u,k) \equiv E(\mathrm{am}\, u,k) \equiv E(\varphi, k)$

$$= D''(t^2 - 1/k^2)^{-1/2} (t^2 - 1)^{-1/2}\, dt$$

$$= D''/k\, (k^2t^2 - 1)^{-1/2} (t^2 - 1)^{-1/2}\, dt$$

Let $\xi = t$ and $D = D''/k$ Then

$$dw = D(1 - k^2\xi^2)^{-1/2}(1 - \xi^2)^{-1/2}\, d\xi \text{ and}$$

$$w = D\int_0^{\varphi} \frac{d\xi}{\sqrt{1 - k^2\xi^2}\,\sqrt{1 - \xi^2}} + N$$

where D and N are constants, possibly complex, which determine the size and position of the w-plane polygon Equation (A-5) is seen to have the same form as (A-1) with solution

$$w = DF(\sin^{-1}t,k) + N \tag{A-6}$$

At point (4) $t = -1$, $w = -K(k)$ and $\varphi = \sin^{-1}(-1)$ giving

$$w = DF(-\pi/2,k) + N \text{ or}$$

$$-K(k) = -DK(k) + N \tag{A-7}$$

At point (5) $t = +1$, $w = +K(k)$ and $\varphi = \sin^{-1}(+1)$ giving

$$w = DF(+\pi/2,k) + N \text{ or}$$

$$+K(k) = +DK(k) + N \tag{A-8}$$

Addition of (A-7) and (A-8) gives $N = 0$ and subtraction $D = 1$. With these values (A-6) becomes

$$w = F(\sin^{-1}t,k) \tag{A-9}$$

Using (A-9) and (A-4), the w to t transformation is seen to be

$$t = \text{sn } w \tag{A-10}$$

Capacitance Per Unit Length and Impedance

The free-space capacitance per unit-length of generalized microstrip can now be obtained directly from the w-plane map as permittivity times conductor width/separation, $c_0/2 = \epsilon_0 K'/2K$ Farads/meter for the upper half plane, or

$$c_0 = \epsilon_0 \frac{K'}{K} \tag{A-11}$$

Farads/meter for the total capacitance, including the lower half plane. The propagation velocity on the microstrip is $v_p = (\mu\epsilon)^{1/2}$. For free space $v_p = (\mu_0\epsilon_0)^{1/2} = c$, the free-space velocity of light. The characteristic impedance of microstrip is defined by $Z = (L/C)^{1/2}$. For free-space $L = \mu_0$ and $C = c_0$, giving $Z_0 = (\mu_0/c_0)^{1/2}$ or

$$Z_0 = \eta \frac{K}{K'} \tag{A-12}$$

with $\eta = \sqrt{\dfrac{\mu_0}{\epsilon_0}} \approx 376.7\ \Omega$

Thus far, expressions for the free-space capacitance and impedance of generalized microstrip have been obtained. It remains to determine the relationship between these parameters and the geometry of the conducting strips by finding the *w* to *z*-plane transformation.

The *w* to *z*-plane Transformation

The finite portion of the z-plane polygon is shown at the top of Fig. A-1. This polygon includes infinity as vertex (1). The Schwarzian differential is obtained by traversing the periphery of the polygon from (1) on the left through (2), (3), (4), (5), (6) and (7) to (1) on the right, then to infinity and back to (1) on the left, turning through an angle of 2π radians between (1) right and (1) left. The transform points are given in Table A-1 and result in Schwarzian derivative,

$$dz/dt = D'(t - C1/k)(t + C2/k)(t + 1/Rk)^{-2}$$
$$\times ((t + 1/k)(t - 1/k)(t + 1)(t - 1))^{-1/2} \qquad \text{(A-13)}$$

The derivative of the w to t-plane transform was derived in the previous section, it is

$$dw/dt = D''((t + 1/k)(t - 1/k)(t + 1)(t - 1))^{-1/2} \qquad \text{(A-14)}$$

Dividing (A-14) by (A-13) yields

$$dz/dw = (dz/dt)/(dw/dt) = D(t - C1/k)(t + C2/k)(t + 1/Rk)^{-2} \qquad \text{(A-15)}$$

where $D = D'/D''$ The left side of this (A 15) is a function of w whereas the right side is a function of t but (A-14) was previously integrated, resulting in the transformation $t = \mathrm{sn}\, w$ Substitution of sn w for t, the dummy variable ξ for w, and integrating both sides of (A-15), gives the integral that must be evaluated

$$z(w) = D\int_0^w \frac{(\mathrm{sn}\,\xi - C1/k)(\mathrm{sn}\,\xi + C2/k)}{(\mathrm{sn}\,\xi + 1/Rk)^2}\, d\xi \qquad \text{(A-16)}$$

Expanding the numerator, (A 16) can be written

$$z(w) = D\left[\int_0^w \frac{\mathrm{sn}^2\,\xi}{(\mathrm{sn}\,\xi + 1/Rk)^2}\, d\xi + \int_0^w \frac{(C2/k - C1/k)\mathrm{sn}\,\xi}{(\mathrm{sn}\,\xi + 1/Rk)^2}\, d\xi - \int_0^w \frac{C1\,C2/k^2}{(\mathrm{sn}\,\xi + 1/Rk)^2}\, d\xi\right]$$

(A-17)

Direct Integration

The first attempt is direct integration of (A-17) term by term. The integral of the third term of (A-17) is found in Byrd and Friedman (B.F. 330.51, 330.50, and 361.58) which are shown below.

B.F. 330.51
$$\alpha_{-2}=\int\frac{du}{(\alpha_1+\operatorname{sn}u)^2}=\frac{1}{(\alpha_1^2-1)(\alpha_1^2k^2-1)}[(2k^2\alpha_1^2-1-k^2)$$
$$\times\Pi(\varphi,1/\alpha_1^2,k)-E(\varphi,k)+(1-k^2\alpha_1^2)F(\varphi,k)$$
$$+\frac{(1+k^2\,2\alpha_1^2k^2)\alpha_{-1}}{\alpha_1}\quad\frac{\operatorname{cn}u\operatorname{dn}u}{\alpha_1(\alpha_1+\operatorname{sn}u)}]$$

B.F. 330.50
$$\alpha_{-1}=\int\frac{du}{\alpha_1+\operatorname{sn}u}=\frac{1}{\alpha_1}[\Pi(\varphi,1/\alpha_1^2,k)-f\alpha_1]$$

B.F. 361.58
$$f=\frac{1}{2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}}\ln[\frac{2(\alpha^2-1)(\alpha^2-k^2)+(1-\alpha^2\operatorname{sn}^2u)(\alpha^2+\alpha^2k^2-2k^2)}{1-\alpha^2\operatorname{sn}^2u}$$
$$+\frac{2\alpha^2\sqrt{(\alpha^2-1)(\alpha^2-k^2)}\operatorname{cn}u\operatorname{dn}u}{1-\alpha^2\operatorname{sn}^2u}],\quad[(1-\alpha^2)(\alpha^2-k^2)<1]$$

This is the hyperbolic case of the elliptic integral of the third kind, since $0\le\alpha^2\le k^2$. The complexity of this integral is such that it appears unlikely that the first two terms can be directly integrated.

Integration of the Reduced Equation

It is noted that the transformation must be of the form

$$z(w)=\Theta(w)\mid_{z=0}+\Psi(w,R)\tag{A-18}$$

where $\Theta(w)$ and $\Psi(R,w)$ are undetermined functions of R and w, and $\Theta(w)$ is the function of w that remains when $R = 0$. With $R = 0$, the transformation reduces to that for equal strips (balanced microstrip). Taking R as zero puts the pole in the t plane $(1/Rk)$ at infinity, thus removing it from the integral. The resulting reduced integral, with $C1 = C2 = C0$, is

$$z(w)\big|_{R=0} = D'\left[\int_0^w \mathrm{sn}^2\xi \, d\xi - (C0/k)^2 \int_0^w d\xi\right] + N \qquad \text{(A-19)}$$

This equation is integrated using B.F. 310.02, yielding

$$\begin{aligned} z(w)\,\big|_{R=0} &= D'[(1/k^2)(w - E(w)) - (C0/k)^2 w)] + N, \\ &= D[(1 - C0^2)w - E(w)] + N \qquad \text{(A-20)} \end{aligned}$$

with $D = D'/k^2$

The constants D, N, and $C0$ are evaluated by setting z and w to appropriate values at points (2) through (7) using Fig. A-1 and Table A-1.

At point (5) $z = +h$, $w = +K$ giving

$$+h = D[+(1 - C0^2)K - E(K)] + N \qquad \text{(A-21)}$$

At point (4) $z = -h$, $w = -K$ giving

$$-h = D[-(1 - C0^2)K + E(K)] + N \qquad \text{(A-22)}$$

Addition of (A-21) and (A-22) gives

$$0 = 0 + 2N, \text{ or } N = 0$$

At point (7) $z = +h$, $w = K + iK'$ and using I-24 to expand $E(K + iK')$, gives,

$$+h = D[(1 - C0^2)K + i(1 - C0^2)K' - E - i(K' - E')] \qquad \text{(A-23)}$$

At point (2) $z = -h$, $w = -K + iK'$, giving

$$-h = D[-(1 - C0^2)K + i(1 - C0^2)K' + E - i(K' - E')] \qquad \text{(A-24)}$$

Adding (A-23) and (A-24) gives

$$0 = i\,2D[(1 - C0^2)K' - K' - E'],$$

$$(1 - C0^2)K' = K' - E',$$

$$1 - C0^2 = 1 - E'/K',$$

$$C0 = \sqrt{\frac{E'}{K'}} \qquad \text{(A-25)}$$

Substitution of (A-25) into (A-20), with $N = 0$, gives

$$z = D\left[(1 - \frac{E'}{K'})w - E(w)\right] \qquad \text{(A-26)}$$

At point (5) $z = +h$, $w = +K$ giving

$$+h = D\left[+(1 - \frac{E'}{K'})K - E\right] \qquad \text{(A-27)}$$

At point (4) $z = -h$, $w = -K$ giving

$$-h = D\left[-(1 - \frac{E'}{K'})K + E\right] \qquad \text{(A-28)}$$

Subtraction of (A-28) from (A-27) gives

$$2h = 2D\left[(1 - \frac{E'}{K'})K - E\right] \text{ or}$$

$$D = \frac{hK'}{KK' - E'K - EK'} \qquad \text{(A-29)}$$

Using 1-26, Legendre's relationship, the denominator of (A-29) is seen to be equal to $-\pi/2$ giving

$$D = \frac{2hK'}{\pi} \qquad \text{(A-30)}$$

Substitution of (A-30) into (A-26) gives the w to z-plane transformation for the special case, $R = 0$,

$$z(w)\big|_{R=0} = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w)\right] \tag{A-31}$$

Integration of the Transformation Integral

The earlier attempt at direct integration of (A-17), term by term, gave such a complicated result that the approach was abandoned. It is unusual for physical or geometric transforms to yield such complicated results, so it is possible that there is cancellation between terms. With this possibility in mind, assume that the integral has the form of (A-18). Substituting (A-31) in (A-18) gives

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) - \Psi(w,R)\right] + N \tag{A-32}$$

where $\Psi(w,R)$ is a function to be determined. The negative sign on $\Psi(w,R)$ is from eq. (A-30). Returning to (A-17), it is noted that the denominator appears as the square of a function. Thus, $d\Psi/dw$ has the form of the derivative of a quotient, given by the well known formula

$$d\left(\frac{U}{V}\right) = \frac{VdU - UdV}{V^2}$$

Taking $\mathrm{U} = \psi(w)$ and $\mathrm{V} = \mathrm{sn}\, w + 1/Rk$ gives

$$\Psi(w,R) = \frac{-\psi(w)}{\mathrm{sn}\, w + 1/Rk} \tag{A-33}$$

with $\psi(w)$ replacing $\Psi(R, w)$ as the unknown function to be determined. Substitution of (A-33) into (A-32) gives

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) - \frac{\psi(w)}{\mathrm{sn}\, w + 1/Rk}\right] + N \tag{A-34}$$

Comparing (A-34) to B.F. 330.51, obtained earlier in the attempt to directly integrate (A-17), shows that the last term of the integral has the same form as (A-33) and that $-\mathrm{cn}\, w\, \mathrm{dn}\, w$ is a likely candidate for $\psi(w)$. Substitution of $-\mathrm{cn}\, w\, \mathrm{dn}\, w$ for $\psi(w)$ gives the w to z plane transformation as

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) + \frac{\mathrm{cn}\, w\, \mathrm{dn}\, w}{\mathrm{sn}\, w + 1/Rk}\right] + N \qquad \text{(A-35)}$$

provided that $\psi(w)$ has been correctly selected. The correctness of (A-35) cannot be directly established by comparison to (A-17).

Determination of Constants

In order to compare (A-35) to (A-17), which is in derivative form, it is necessary to take the derivative of (A-35), equate it to (A-17) and determine the constants $C1$ and $C2$ in terms of the coefficients of (A-35). The derivative of (A-35) is obtained using B.F. 314.02, 731.01, 731.02 and 731.03 which gives

$$\frac{dz}{dw} = \frac{2hK'}{\pi}\left[\frac{E'}{K'} - 1 + d^2 + \frac{(s + 1/Rk)(-k^2sc^2 - sd^2) - c^2d^2}{(\mathrm{sn}\, w + 1/Rk)^2}\right] + N \qquad \text{(A-36)}$$

To simplify notation and manipulation let $L = 2hK'/\pi$, $M = E'/K' - 1$ and $\xi = 1/Rk$. Making the indicated substitutions and clearing fractions gives

$$\frac{dz}{dw} = L\left[\frac{(s+\xi)^2M + (s+\xi)^2d^2 - (s+\xi)(k^2sc^2 + sd^2) - c^2d^2}{(s+\xi)^2}\right] \qquad \text{(A-37)}$$

Taking the derivative of both sides of (A-17) gives

$$\frac{dz}{dw} = D\left[\frac{s^2 + (C2/k - C1/k)s - C1\,C2/k^2}{(s+\xi)^2}\right] \qquad \text{(A-38)}$$

Equating (A-37) and (A-38) allows cancellation of the denominators, giving

$$L[(s + \xi)^2M + (s + \xi)^2d^2 - (s + \xi)(k^2sc^2 + sd^2) - c^2d^2]$$

$$= D[s^2 + (C2/k - C1/k)s - C1C2/k^2] \qquad \text{(A-39)}$$

To establish the equality (A-39) the two sides must be in the same form; this is done by manipulating the left hand side to obtain

$$\text{LHS} = L[coeff_1 \times \text{sn}^2\, w + coeff_2 \times \text{sn}\, w + coeff_3\,] \qquad \text{(A-40)}$$

Obtaining the form, (A-40) is a necessary condition for (A-35) to be a solution. In the following steps the LHS is manipulated using identities I-6 through I-10 from Table A-2.

$$\begin{aligned}
\text{LHS} &= L[(s + \xi)^2 M + (s + \xi)^2 d^2 - k^2 s^2 c^2 - s^2 d^2 - \xi s k^2 c^2 - \xi s d^2 - c^2 d^2] \\
&= L[(s + \xi)^2 M + (s + \xi)^2 d^2 - k^2 s^2 c^2 - (c^2 + s^2) d^2 - \xi s k^2 c^2 - \xi s d^2] \\
&= L[(s + \xi)^2 M + s^2 d^2 + 2\xi s d^2 + \xi^2 d^2 - k^2 s^2 c^2 - d^2 - \xi s k^2 c^2 - \xi s d^2] \\
&= L[(s + \xi)^2 M + s^2 d^2 - k^2 s^2 c^2 + \xi s d^2 - \xi s k^2 c^2 + \xi^2 d^2 - d^2] \\
&= L[(s + \xi)^2 M + k'^2 s^2 + k'^2 \xi s + \xi^2 d^2 - d^2] \\
&= L[(s + \xi)^2 M + (1 - k^2) s^2 + (1 - k^2)\xi s + (\xi^2 - 1) d^2] \\
&= L[(s + \xi)^2 M + s^2 - k^2 s^2 + (1 - k^2)\xi s + (\xi^2 - 1)(1 - k^2 s^2)] \\
&= L[(s + \xi)^2 M + s^2 - k^2 s^2 + (1 - k^2)\xi s + \xi^2 - k^2 \xi^2 s^2 - 1 + k^2 s^2] \\
&= L[s^2 M + 2M\xi s + M\xi^2 + s^2 + (1 - k^2)\xi s + \xi^2 - k^2 \xi^2 s^2 - 1] \\
&= L[(1 + M - k^2\xi^2) s^2 + (1 + 2M - k^2)\xi s + (1 + M)\xi^2 - 1] \qquad \text{(A-41)}
\end{aligned}$$

Substitution of (A-41) for the LHS of (A-39) gives

$$\begin{aligned}
&L[(1 + M - k^2\xi^2) s^2 + (1 + 2M - k^2)\xi s + (1 + M)\xi^2 - 1] \\
&\qquad = D[s^2 + (C2/k - C1/k)s - C1C2/k^2] \qquad \text{(A-42)}
\end{aligned}$$

The equality (A-42) holds if

$$D = L(1 + M - k^2\xi^2), \qquad \text{(A-43)}$$

$$D(C2 - C1)/k = L(1 + 2M - k^2)\xi \text{ and} \qquad \text{(A-44)}$$

$$- DC1C2/k^2 = L(\xi^2 + M\xi^2 - 1) \qquad \text{(A-45)}$$

which can be equivalently written

$$L = D/(1 + M - k^2\xi^2),$$

$$(C2 - C1) = (1 + 2M - k^2)k\xi/(1 + M - k^2\xi^2), \text{ and} \qquad \text{(A-46)}$$

$$C1C2 = (1 - \xi^2 - M\xi^2)k^2/(1 + M - k^2\xi^2). \qquad \text{(A-47)}$$

Substitution of $M = E'/K' - 1$ and $\xi = 1/Rk$ into (A-46), and (A-47), and after some manipulation, the constants can be written

$$C2 - C1 = \frac{R(2E' - K' - k^2K')}{R^2E' - K'}, \qquad \text{(A-48)}$$

$$C1 \times C2 = \frac{R^2k^2K' - E'}{R^2E' - K'} \qquad \text{(A-49)}$$

Back substitution of (A-43), (A-44) and (A-45) into the LHS of (A-42) establishes the equivalence of the derivatives and validity of the derived w to z-plane transformation, equation (A 35), rewritten here for reference

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) + \frac{\operatorname{cn} w \operatorname{dn} w}{\operatorname{sn} w + 1/Rk}\right] + N \qquad \text{(A-50)}$$

Evaluation of the constant N completes the derivation of the transform function

The constant N is determined by evaluation of (A-50) at corresponding points of the z and w-planes, using Fig A-1, Table A-1.

At point (5) $z = +h + i0$ and $w = K$ giving

$$+h = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)K + E(K) + \frac{\operatorname{cn} K \operatorname{dn} K}{\operatorname{sn} K + 1/Rk}\right] + N \qquad \text{(A-51)}$$

At point (4) $z = -h + i0$ and $w = -K$ giving

$$-k = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)(-K) + E(-K) + \frac{\mathrm{cn}\,-K\,\mathrm{dn}\,-K}{\mathrm{sn}\,-K + 1/Rk}\right] + N \qquad \text{(A-52)}$$

Using identities 1-12 to 14, 1-17 to 19 and 1-29 in (A-51) and (A-52) gives

$$+k = \frac{2hK'}{\pi}\left[+\left(\frac{E'}{K'} - 1\right)K(k) + E(k)\right] + N, \qquad \text{(A-53)}$$

$$-k = \frac{2hK'}{\pi}\left[-\left(\frac{E'}{K'} - 1\right)K(k) - E(k)\right] + N \qquad \text{(A-54)}$$

Adding (A-53) and (A-54) gives $0 = 2N$ or $N = 0$. This completes the derivation of the w to z-transformation which can now be written in final form as

$$z(w) = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)w + E(w) + \frac{\mathrm{cn}\,w\,\mathrm{dn}\,w}{\mathrm{sn}\,w + 1/Rk}\right] \qquad \text{(A-55)}$$

Reduction of the Transformation to Real Form

Equation (A-55) is in complex form with $z = x + iy$ and $w = u + iv$, a form that is not computationally useful. It can be reduced to real form by expanding its terms, using formulas 1-21, 1-23, 1-25 and 1-27, then equating x to the real part of the right hand side of (A-55) and y to the imaginary part. Direct application of this procedure results in a rather complicated and cumbersome expression on the right hand side but can be simplified somewhat using the notation of (A-18). With this procedure the equations for $x(u,v)|_{R=0} = \Re[\Theta(u,v)]$ and $y(u,v)|_{R=0} = \Im[\Theta(u,v)]$ where $\Re$ and $\Im$ designate the real and imaginary parts respectively. The general transformations are then obtained by adding $\Re[\Psi(u,v,R)]$ and $\Im[\Psi(u,v,R)]$ to the result obtained for $R = 0$. This gives $x(u,v,R) = \Re[\Theta(u,v)] + \Re[\Psi(u,v,R)]$ and $y(u,v,R) = \Im[\Theta(u,v)] + \Im[\Psi(u,v,R)]$.

Beginning with (A-31) and using formula 1-27 gives

$$x(u,v)\big|_{R=0} = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)u + E(u,k) + \frac{k^2 scds_1^2}{\Delta}\right] \qquad \text{(A-56)}$$

and
$$y(u,v)\big|_{R=0} = \frac{2hK'}{\pi}\left[\frac{E'}{K'}v - E(v,k') + \frac{s_1 c_1 d_1 d^2}{\Delta}\right] \qquad \text{(A-57)}$$

Equation (A-56) can be written in a somewhat better form as follows

$$x(u,v)\big|_{R=0} = \frac{2hK'}{\pi}\left[\frac{E'K - KK'}{KK'}u + E(u,k) + \frac{k^2 scds_1^2}{\Delta}\right] \qquad \text{(A-58)}$$

Using I-30 gives $E'K - KK' = \pi/2 - EK'$, and substitution into (A-58) gives

$$x(u,v)\big|_{R=0} = \frac{uh}{K} + \frac{2hK'}{\pi}\left[E(u,k) - \frac{E}{K}u + \frac{k^2 scds_1^2}{\Delta}\right] \qquad \text{(A-59)}$$

Separation of $\Psi(u,v,R)$ into its real and imaginary parts is somewhat more tedious and the result significantly more complicated. Using I-21, I-23 and I-25 gives

$$\Re[\Psi(u,v,R)] = \frac{2hK'}{\pi}\left[\frac{R^2k^2[RkA + (AC - BD)/\Delta]}{(RkC + \Delta)^2 + (RkD)^2}\right] \qquad \text{(A-60)}$$

and
$$\Im[\Psi(u,v,R)] = -\frac{2hK'}{\pi}\left[\frac{R^2k^2[RkB + (AD - BC)/\Delta]}{(RkC + \Delta)^2 + (RkD)^2}\right] \qquad \text{(A-61)}$$

where

$A = cdd_1(c_1^2 - k^2s^2s_1^2)$, $B = ss_1c_1(k^2c^2 + d^2d_1^2)$, $C = sd_1$, $D = cds_1c_1$, $\Delta = 1 - d^2s_1^2$, with $s = \mathrm{sn}(u,k)$, $c = \mathrm{cn}(u,k)$, $d = \mathrm{dn}(u,k)$, $s_1 = \mathrm{sn}(v,k')$, $c_1 = \mathrm{cn}(v,k')$, and $d_1 = \mathrm{dn}(v,k')$

It is generally convenient to specify x and y in normalized form as $X = x/h$ and $Y = y/h$. Expressing (A-57) and (A-59) in this fashion, for the special case $R = 0$, gives

$$X = x/h\big|_{R=0} = \frac{u}{K} + \frac{2K'}{\pi}\left[E(u,k) - \frac{E}{K}u + \frac{k^2 scds_1^2}{\Delta}\right], \qquad \text{(A-62)}$$

$$Y = y/h\,|_{R=0} = \frac{2K'}{\pi}\left[\frac{E'}{K'}v - E(v,k') + \frac{s_1c_1d_1d^2}{\Delta}\right] \tag{A-63}$$

The general transformation is given by

$$X = x/h = \frac{u}{K} + \frac{2K'}{\pi}\left[E(u,k) - \frac{E}{K}u + \frac{k^2scds_1^2}{\Delta} + \frac{R^2k^2[RkA + (AC - BD)/\Delta]}{(RkC + \Delta)^2 + (RkD)^2}\right], \tag{A-64}$$

$$Y = y/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}v - E(v,k') + \frac{s_1c_1d_1d^2}{\Delta} - \frac{R^2k^2[RkB + (AD - BC)/\Delta]}{(RkC + \Delta)^2 + (RkD)^2}\right] \tag{A-65}$$

Inverse Transformations

The possible existence of inverse transformations of the form, $u = \xi_1(x,y)$ and $v = \xi_2(x,y)$, was briefly discussed in section A1.0. Unfortunately, for generalized microstrip, no such inverse exists and inversion must be carried out iteratively. For the special case $R = 0$, inverse transformations can be obtained in useful form, though not in the ideal form shown above. The derivable transformations are $v = v(x,u,k)$ and $u = u(y,v,k')$. The generality of these transformations is somewhat limited by the appearance of u as an independent variable of v and v as an independent variable of u. Despite this lack of generality, they do provide a means of directly mapping the dielectric boundaries from the z plane to the w plane. The derivation of $v = v(x,u,k)$ begins with equation (A-62), which can be written

$$\frac{\pi}{2K'}\left[X - \frac{u}{K}\right] - E(u,k) + \frac{E}{K}u = \frac{k^2scds_1^2}{1 - d^2s_1^2} \tag{A-66}$$

Let $P = \frac{\pi}{2K'}\left[X - \frac{u}{K}\right] - E(u,k) + \frac{E}{K}u$, substitute P into (A-66) and solve for s_1^2,

$$\text{giving}\quad s_1^2 = \frac{P}{Pd^2 + k^2scd} \quad \text{and} \tag{A-67}$$

$$c_1^2 = 1 - s_1^2 = \frac{k^2scd - Pk^2s^2}{Pd^2 + k^2scd} \tag{A-68}$$

Let

$$T_x^2 = \mathrm{tn}_1^2 = \frac{s_1^2}{c_1^2} = \frac{1}{k^2}\left[\frac{P}{scd - Ps^2}\right], \quad \text{then} \tag{A-69}$$

$$T_x = \frac{1}{k}\sqrt{\frac{P}{scd - Ps^2}} \quad \text{and} \tag{A-70}$$

$$v(X,u,k) = F(T_x,k'), \tag{A-71}$$

$$\text{with}\quad P = \frac{\pi}{2K'}\left[X - \frac{u}{K}\right] - E(T_x,k) + \frac{E}{K}u \tag{A-72}$$

$$\text{and}\quad T_x = \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u} \tag{A-73}$$

The inverse transformation $u = u(Y,v,k')$ is obtained from equation (A-63) by solving for d^2, giving

$$T_y = \sqrt{\frac{s_1c_1d_1 - Qc_1^2}{Qd_1^2 - k^2s_1c_1d_1}} \quad \text{and} \tag{A-74}$$

$$u(Y,v,k') = F(T_y,k), \tag{A-75}$$

$$\text{with}\quad Q = \frac{\pi}{2K'}Y + E(T_v,k') - \frac{E'}{K'}v \quad \text{and} \tag{A-76}$$

$$T_v = \frac{\mathrm{sn}\,v}{\mathrm{cn}\,v} \tag{A-77}$$

Derivation of Expressions for *a/h* and *b/h*

The derivation of the expression for a/h is obtained by evaluating (A-55) at point (6), where from Table A-1, $z = h + ia$, $w = K + ia'$ and $t = CI/k$

Substitution of these values for z, and w in (A-55) gives

$$h + ia = \frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)(K+ia') + E(K+ia') + \frac{\mathrm{cn}(K+ia')\,\mathrm{dn}(K+ia')}{\mathrm{sn}(K+ia') + 1/Rk}\right] \tag{A-78}$$

From (A-78), it is seen that

$$a = \Im\,\frac{2hK'}{\pi}\left[\left(\frac{E'}{K'} - 1\right)(K+ia') + E(K+ia') + \frac{\mathrm{cn}(K+ia')\,\mathrm{dn}(K+ia')}{\mathrm{sn}(K+ia') + 1/Rk}\right] \tag{A-79}$$

From (A-10), $t = \mathrm{sn}\, w$; at point (6), $t = CI/k$ and $w = K + ia'$ giving

$$CI/k = \mathrm{sn}(K+ia') \tag{A-80}$$

Using formula 1-21, with $u = K$, $v = a'$, and after some manipulation, it is found that $\mathrm{sn}(K + ia') = 1/\mathrm{dn}(a',k')$ giving $CI/k = 1/\mathrm{dn}(a',k')$ or

$$\mathrm{dn}(a',k') = \frac{k}{CI} \tag{A-81}$$

Identity 1-8 gives an expression for $\mathrm{sn}(a',k')$ in terms of $\mathrm{dn}(a',k')$ which can be written

$$\begin{aligned}\mathrm{sn}(a',k') &= \sqrt{\frac{1 - (k/CI)^2}{k'^2}}\\ &= \sqrt{\frac{CI^2 - k^2}{k'^2 CI^2}}\end{aligned} \tag{A-82}$$

Similarly, using identity 1-9, $\mathrm{cn}(a',k')$ can be expressed in terms of $\mathrm{dn}(a',k')$ as

$$\mathrm{cn}(a',k') = k\sqrt{\frac{1 - CI^2}{k'^2 CI^2}} \tag{A-83}$$

Reduction of $\mathrm{cn}(K + ia')$ and $\mathrm{dn}(K + ia')$ is done through use of identities I-23 and I-25, giving

$$\mathrm{cn}(K + ia') = -i\left[\frac{k'\mathrm{sn}(a',k')}{\mathrm{dn}(a',k')}\right] \text{ and} \tag{A-84}$$

$$\mathrm{dn}(K + ia') = \frac{k'\mathrm{cn}(a',k')}{\mathrm{dn}(a',k')} \tag{A-85}$$

Substitution of (A-81) and (A-82) into the right hand side of (A-84) gives

$$\mathrm{cn}(K + ia') = -i\,\frac{\sqrt{CI^2 - k^2}}{k} \tag{A-86}$$

Similarly substitution of (A-81) and (A-83) into the right hand side of (A-85) gives

$$\mathrm{dn}(K + ia') = \sqrt{1 - CI^2} \tag{A-87}$$

Substitution of (A-80), (A-86) and (A-87) into the last term of (A-78) gives

$$\frac{\mathrm{cn}(K + ia')\mathrm{dn}(K + ia')}{\mathrm{sn}(K + ia') + 1/Rk} = -i\left[\frac{\sqrt{CI^2 - k^2}\sqrt{1 - CI^2}}{CI + 1/R}\right] \tag{A-88}$$

The imaginary part of the first two terms of (A-78) is given by

$$\Im\left[\left(\frac{E'}{K'} - 1\right)(K + ia')\right] = i\left(\frac{E'}{K'} - 1\right)a' + \Im E(K + ia') \tag{A-89}$$

The imaginary part of $E(K + ia')$ is evaluated using identity I-27 which gives

$$\Im E(K + ia') = a' - E(a',k') + \frac{k'^2\mathrm{sn}K\mathrm{cn}K}{\mathrm{dn}K} \tag{A-90}$$

From I-18, $\mathrm{cn}\,K = 0$, giving

$$\Im E(K + ia') = a' - E(a',k') \tag{A-91}$$

Substitution of (A-91) into the right hand side of (A-89) gives

$$\Im\left[\left(\frac{E'}{K'} - 1\right)(K + ia') + E(K + ia')\right] = i\left[\left(\frac{E'}{K'} - 1\right)a' + a' - E(a',k')\right]$$

$$= \left[\left(\frac{E'}{K'} \right) a - E(a',k') \right] \qquad \text{(A-92)}$$

Substitution of (A-92) and the right hand side of (A-88) into (A-78) gives the expression for a/h as

$$a/h = \frac{2K'}{\pi} \left[\frac{E'}{K'} a' - E(a',k') - \sqrt{\frac{(CI^2 - k^2)(1 - CI^2)}{CI + 1/R}} \right] \qquad \text{(A-93)}$$

In (A-93), am $a' = \varphi$ is the amplitude and k' is the modulus of the elliptic integrals. Accordingly, (A-93) can be rewritten in Legendre format using I-4, with $v = a' = F(\varphi,k')$, giving

$$a/h = \frac{2K'}{\pi} \left[\frac{E'}{K'} F(\varphi,k') - E(\varphi,k') - \frac{\sqrt{(CI^2 - k^2)(1 - CI^2)}}{CI + 1/R} \right], \qquad \text{(A-94)}$$

$$\varphi = \arcsin[\mathrm{sn}(a',k')] = \arcsin \frac{\sqrt{(CI^2 - k^2)}}{kCI} \qquad \text{(A-95)}$$

Using equations (A-94) and (A-95) a/h can be evaluated. The equations, however, are not numerically stable for φ very near $\pi/2$. A better evaluation of the elliptic integrals is obtained using the Bulirsch algorithms [17]. The upper limit of integration for these algorithms is $T_1 = \tan\varphi = \mathrm{tn}(a',k') = \mathrm{sn}(a',k')/\mathrm{cn}(a',k')$. The $\mathrm{tn}(a',k')$ can be written using (A-82) and (A-83) as

$$T_1 = \frac{\mathrm{sn}(a',k')}{\mathrm{cn}(a',k')} = \frac{1}{k} \sqrt{\frac{CI^2 - k^2}{1 - CI^2}} \qquad \text{(A-96)}$$

$$\text{and } a' = F(T_1,k'), \qquad \text{(A-97)}$$

$$\text{with } [K'/2 \le a' \le K']$$

Substitution of (A-96) into (A-94), and with some manipulation gives

$$a/h = \frac{2K'}{\pi} \left[\frac{E'}{K'} F(T_1,k') - E(T_1,k') + T_1 k (1 - CI^2) \left(\frac{1}{CI} - \frac{R}{1 + RCI} \right) \right] \qquad \text{(A-98)}$$

By a similar procedure the expression for b/h is obtained as

$$T_2 = \frac{\mathrm{sn}(b',k')}{\mathrm{cn}(b',k')} = \frac{1}{k}\sqrt{\frac{C2^2 - k^2}{1 - C2^2}} \tag{A-99}$$

$$\text{and } b' = F(T_2,k'), \tag{A-100}$$

with $[K'/2 \leq b' \leq K']$ and $[b' \geq a]$

$$b/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}F(T_2,k') - E(T_2,k') + T_2k(1 - C2^2)\left(\frac{1}{C2} + \frac{R}{1 - RC2}\right)\right] \tag{A-101}$$

Expression for $a/h = b/h$ with $R = 0.0$

For the special case of balanced microstrip, $R = 0\ 0$, the formulas are

$$C0^2 = \frac{E'}{K'}, \tag{A-102}$$

$$T_0 = \frac{1}{k}\sqrt{\frac{C0^2 - k^2}{1 - C0^2}}, \tag{A-103}$$

$$a'|_{R=0} = b' = F(T_0,k') \text{ and} \tag{A-104}$$

$$a/h|_{R=0} = b/h = \frac{2K'}{\pi}\left[\frac{E'}{K'}F(T_0,k') - E(T_0,k') + \frac{T_0k(1 - C0^2)}{C0}\right] \tag{A-105}$$

Derivation of the Expression for u_0

The final parameter to be evaluated is u_0, $[-K \leq u_0 \leq 0]$ At point (1),

$t = -1/Rk$

and $w = u_0 + iK'$ with $t = \mathrm{sn}\, w$ giving

$$-1/Rk = \mathrm{sn}(u_0 + iK'). \tag{A-106}$$

The right hand side of (A 105) is evaluated using 1 21, giving

$$\mathrm{sn}(u_0 + iK') = 1/(k\mathrm{sn}(u_0,k) \tag{A 107}$$

Substitution of (A-106) into (A-105) gives

$$-1/Rk = 1/(k\,\mathrm{sn}(u_0,k) \text{ or}$$

$$-R = \mathrm{sn}(u_0,k) \qquad \text{(A-108)}$$

Squaring both sides of (A-107) gives $\mathrm{sn}^2(u_0,k) = R^2$, so $\mathrm{cn}^2(u_0,k) = 1 - R^2$ and $\mathrm{tn}^2(u_0,k) = \mathrm{sn}^2(u_0,k)/\mathrm{cn}^2(u_0,k)$, which can be written

$$T_3 = \mathrm{tn}(u_0,k) = -\sqrt{\frac{R^2}{1-R^2}} \qquad \text{(A-109)}$$

$$u_0 = F(T_3,k), \qquad \text{(A-110)}$$

with $-K \le u_0 \le 0$

This completes the derivation of equations for the free-space parameters of generalized microstrip

APPENDIX B
ANALYSIS OF THE DIELECTRIC POLARIZATION CAPACITANCE

Conformal transformation

The analysis of this appendix is exact for the dielectric polarization due to electrification of the inner faces of the conducting strips.

Figure B-1 is a diagram of the transformations The polygons representing space coordinates (z-plane) and flux potential coordinates (w plane) are mapped to two half planes using the well known transformation $t = \mathrm{sn}(z',k)$ and $t' = \mathrm{sn}(w,p)$ Those unfamiliar with this transformation may refer to the section A 2 1 of Appendix A where the transformation is derived These polygons are of dimensions N by N' and P by P' where N, N', P and P' are complete elliptic integrals of the first kind with moduli k and p, respectively The z and z'-planes are connected by the scale factor $m = N/n = N'/n'$

The intermediate half-planes are connected by the bilinear transformation

$$t' = \frac{(f-d)(t-c)}{(d-c)(f-t)} \tag{B-1}$$

At point (5), $t' = 1/p$, $t = e$, $c = 0$ and $f = \infty$ Substitution of these values in (B-1) gives

$$p = d/e \tag{B-2}$$

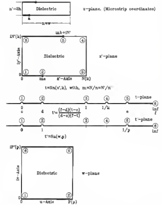

Fig. B-1 Transformation diagram for z, z', t, t', and w-planes for determination of the polarization capacitance of generalized microstrip.

At point (2), $d = \mathrm{sn}(ma)$. At point (5), $e = \mathrm{sn}(mb + iN') = 1/[k\mathrm{sn}(mb)]$, using 1-21 of Table A-2. Substitution for d and e in (B-2) gives

$$p = d/e = k\mathrm{sn}(ma,k)\mathrm{sn}(mb,k) \tag{B-3}$$

The complete elliptic integral, N, is the width of the dielectric sheet and N' is its thickness, with ma the width of the narrow conductor and mb the width of the wide conductor and $m = N'/n' = N'/2h$. Then

$$ma = \frac{N'a}{2h} \text{ and } mb = \frac{N'b}{2h}$$

At this point it is convenient to introduce the assumption that the width of dielectric sheet is very much greater than its thickness, $2h$, and the width of the narrow conductor, a. This assumption gives results that are independent of dielectric width. If the dielectric is narrow, the exact formulation can be used.

In the limit as $N \rightarrow \infty$, $N' = \pi/2$ and $k = 1$. With substitutions for ma, mb and k (B-3) becomes

$$p = \mathrm{sn}\left(\frac{\pi a}{4h}, 1\right)\mathrm{sn}\left(\frac{\pi b}{4h}, 1\right) \tag{B-4}$$

By B.F. 122.09, $\mathrm{sn}(u,1) = \tanh(u)$, yielding

$$p = \tanh\left(\frac{\pi a}{4h}\right)\tanh\left(\frac{\pi b}{4h}\right) \tag{B-5}$$

Equation (B-5) is a simple formulation for finding the elliptic modulus, p, but it may fail numerically at low impedance where p approaches 1. This can be overcome by using the expression for p' which is obtained by substitution of $\tanh^2 u = 1 - \mathrm{sech}^2 u$ and $p^2 = 1 - p'^2$ in (B5), giving

$$p' = \sqrt{\mathrm{sech}^2 A + \mathrm{sech}^2 B - \mathrm{sech}^2 A\,\mathrm{sech}^2 B} \tag{B-6}$$

where $A = \pi a/4h$, and $B = \pi b/4h$

The geometric capacitance, computed as dielectric constant times conductor width divided by conductor separation in the w plane of Fig. B-1, is

$$c_d = \epsilon_0 \epsilon_r \frac{2P(p)}{P'(p)} \tag{B-7}$$

Bound and Unbound Charge

Equation (B-7) is the total internal capacitance of the microstrip in the z-plane of Fig. B-1. It includes not only the capacitance due to polarization of the dielectric but also the capacitance due to free charges on the conductors. Because of this, direct use of (B-7) as the polarization capacitance would overstate the capacitance by the fraction of capacitance attributable to free charge.

Dielectric material between the conductors of microstrip increases the capacitance by polarizing charges at the conductor faces that would otherwise contribute to the internal electric field. The phenomena of polarization can be visualized as the binding together of dipoles to form a dipole chain under the influence of the applied potential with their free ends terminating on the conductor surfaces. These dipole chains constitute the polarization field and do not contribute to the electric field. The voltage across a microstrip transmission line can be written

$$V = \frac{Q}{\epsilon_r} \frac{1}{c_0}, \tag{B-8}$$

where c_0 is the geometric capacitance. Equation (B-8) states that only $1/\epsilon_r$ of the total charge Q contributes to the electric field between the conductors. Thus fraction is called the unbound charge, and can be written

$$q_0 = \frac{Q}{\epsilon_r} \tag{B-9}$$

The remainder,

$$q_p = Q(1 - \frac{1}{\epsilon_r}), \tag{B-10}$$

called the bound charge, is neutralized by the dielectric and therefore contributes nothing to the electric field within the microstrip. Capacitance is defined as $C = Q/V$, so using (B-10) and (B-7) it is seen that the polarization capacitance is

$$\begin{aligned} c_p &= \epsilon_0 \epsilon_r \frac{2P(p)}{P'(p)}(1 - \frac{1}{\epsilon_r}), \\ &= \epsilon_0 \frac{2P(p)}{P'(p)}(\epsilon_r - 1) \end{aligned} \tag{B-11}$$

APPENDIX C
PROGRAM SOURCE CODE LISTINGS

```
C         PROGRAM MIST1.FOR
C PURPOSE
C      Finds the design parameters of generalized microstrip using
C      the exact formulation of eq (3-23).  Given the free-space
C      characteristic impedance Z0, the relative dielectric constant
C      and the parameter R, this program computes the characteristic
C      impedance on a dielectric sheet (Z), the spatial parameters
C      a/h and b/h, the relative phase velocity, and the microstrip
C      capacitance per meter in pico farads.  It simultaneously
C      computes these same values using the approximate equation
C      (3-28).  This allows comparison of the approximation to the
C      exact formulation for the purpose of validating the
C      approximation.  This program is not intended for design
C      calculations but only for validation of approximate
C      equations, which are much less computationally intensive
C      and thus better suited for design applications.
C USAGE
C      The program is interactive asking first for impedance  Z
C      then the parameter R and the relative dielectric constant.
C      Respond with the desired free space impedance, between 13.5
C      and 840.0 Ohms, parameter R from 0.  to 1.0 and relative
C      dielectric constant greater than or equal to one.  The program
C      returns microstrip parameters on a dielectric sheet Z, a/h, vp
C      and cap/m then returns to request impedance Z0.  Entering 0.0
C      for impedance exits the program.  Entering -1.0 for R or 0.0
C      for dielectric constant returns to the request for Z.
C DESCRIPTION OF PARAMETERS
C      Z0    - FREE-SPACE MICROSTRIP IMPEDANCE
C      Z     - MICROSTRIP IMPEDANCE ON A DIELECTRIC SHEET
C      R     - A PARAMETER RELATED TO THE RATIO a/b
C      EPS   - RELATIVE DIELECTRIC CONSTANT OF THE SHEET
C      a/h   - RATIO OF NARROW CONDUCTOR WIDTH TO SEPARATION
C      b/h   - RATIO OF WIDE CONDUCTOR WIDTH TO SEPARATION
C      vp    - PHASE VELOCITY RELATIVE TO THE FREE-SPACE VELOCITY OF
```

```
C             LIGHT
C     cap/m  CAPACITANCE IN PICO FARADS PER METER
C REMARKS
C     Valid for 13.0 < Z < 840.0 Ohms, and EPS equal to or greater
C     than 1.0
C     Accuracy
C     For FREE-SPACE impedance between 13.0 and 840.0 Ohms, the
C     calculation exactly follows eq (3-23), with an absolute accuracy
C     of approximately 0.001*Z, including refraction.
C
C     The accuracy of the approximate equation (3-28) is better than
C     0.25 % of Z as follows
C          Error         Z0        R      diel  const
C      -0.0024445*Z   70.602   0.766       2.292
C       0.0024386*Z  278.400   0.580       2.393
C     The accuracy of another approximate equation () was also
C     checked and found to be less than 0.75 % of Z as follows
C          Error         Z0        R      diel  const
C       0.007255*Z   130.000   0.820       2.308
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DAREA,DCEL2,DCONST,DCP,DEL0,DEL2,DFINDU,DFINDV,DPARAM,DQG32,
C     DSCD,DXOFU,DXOFV,DZERO
C METHOD
C     The program computes the characteristic impedance using (3-27)
C     and effective dielectric constant using (3-23).  Numerical
C     integration is used to find the dielectric area (CFMAX)
C REFERENCE
C     A. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL
C     1990.
C-------------------------------------------------------------------
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      PARAMETER(CC=2.99792458D0)
C
10    WRITE(5,1)
      READ(*,3) Z0
      IF(Z0.LE.0.D0) GOTO 500
      CALL DEL0(Z0)
      IF(XK*XK.LT.1.D-40) THEN
        WRITE(5,4)
        GOTO 10
      ENDIF
      WRITE(5,2)
      READ(*,3) R
      IF(R.LT.0.D0.OR.R.GT.1.D0) GOTO 10
      CALL DCONST(R,C1,C2)
      IF(C1*C1 - KK.LT.1.D-8) THEN
        WRITE(5,5)
        GOTO 10
      ENDIF
      CALL DPARAM(R,C1,C2,AP,BP,U0,AH,BH)
      CALL DCP(AH,BH,CP)
      CALL DAREA(R,AP,U0,CPMAX)
20    WRITE(5,6)
      READ(*,3) REPS
      IF(REPS.LT.1.D0) GOTO 10
C Exact formulation of effective dielectric constant eq.(3.23)
      CI=CP*(REPS-1.D0)+AP
      CE=(1.D0-AP)/(1.D0-(1.D0-1.D0/REPS)*(CPMAX-CP)/(1.D0-AP))
      EFFK=CI+CE
      VP=1.D0/DSQRT(EFFK)
      Z=Z0*VP
      CAP=1.D4/(CC*VP*Z)
      WRITE(5,12)
      WRITE(5,7)
      WRITE(5,8) Z,AH,BH,VP,CAP
C Approximate formulation of effective dielectric constant eq.(3.28)
      CI=CP*(REPS-1.D0)+AP
      CE=(1.D0-AP)/(1.D0-0.536D0*(1.D0-1.D0/REPS)*(CP-AP)/(1.D0-AP))
      EFFK=CI+CE
      VP=1.D0/DSQRT(EFFK)
      Z1=Z0*VP
      CAP=1.D4/(CC*VP*Z1)
      WRITE(5,9)
      WRITE(5,7)
      WRITE(5,8) Z1,AH,BH,VP,CAP
      WRITE(5,11) 100.D0*(Z-Z1)/Z
      WRITE(5,13)
C                        FORMAT STATEMENTS
1     FORMAT(//' Type [13.5 LE Z0 LE 840.0], [0.0] to quit  Z0 = ',$)
2     FORMAT(' Type [0.0 LE R LE 1.0], [-1.0] to change Z0  R = ',$)
3     FORMAT(D25.16)
4     FORMAT(16X'IMPEDANCE TOO LOW, REENTER Z0')
5     FORMAT(16X'IMPEDANCE TOO HIGH, REENTER Z0')
6     FORMAT(' Type EPS, [EPS GE 1.0], [0.0] for new Z    EPS = ',$)
```

```
7      FORMAT(6X,'Z',14X,'a/h',13X,'b/h',13X 'Vp' 11X,'cap pf/m')
8      FORMAT(9G16 8)
9      FORMAT(/33X'APPROXIMATION')
11     FORMAT(/10X' DIFF % = 100 *(EXACT APPROX)/EXACT = ' F18 15)
12     FORMAT(/30X 'EXACT FORMULATION')
13     FORMAT(/)
       GOTO 20
500    STOP
       END

C          PROGRAM MIST2 FOR
C PURPOSE
C      Finds the design parameters of UNBALANCED microstrip (R = 1 0)
C      or BALANCED microstrip (R = 0 0)   Given the desired
C      characteristic impedance  Z  and the relative dielectric
C      constant  this program computes the spatial parameter a/h,
c      the relative phase velocity  and the capacitance per meter in
C      pico farads
C USAGE
C      This program is interactive asking first for impedance, Z
C      then relative dielectric constant   Respond with the desired
C      impedance *  It will next ask for relative dielectric constant
C      which must be greater than or equal to one   The program
C      returns Z, a/h, vp  and cap/m  returning to request impedance
C      Z   Entering 0  for impedance impedance exits the program
C      Entering 0  for dielectric constant returns to request for Z
C
C    * The acceptable values of Z vary with dielectric constant  but
C      for free space are
C      Unbalanced -  6 75 < Z <  634 5
C      Balanced   - 13 50 < Z < 1269 0
C      This range exceeds the reasonable range of microstrip impedance
C      so no difficulty will result for practical calculations   A
C      warning message appears if impedance is out of range and the
C      program returns to the request for Z
C DESCRIPTION OF PARAMETERS
C      Z     - DESIRED MICROSTRIP IMPEDANCE ON A DIELECTRIC SHEET
C      REPS  - RELATIVE DIELECTRIC CONSTANT OF THE SHEET
C      a/h   - RATIO OF CONDUCTOR WIDTH TO SEPARATION
C      b/h   - RATIO OF CONDUCTOR WIDTH TO SEPARATION (With a/h=b/h)
C      vp    - PHASE VELOCITY RELATIVE TO THE FREE SPACE VELOCITY OF
C              LIGHT
C      cap/m- CAPACITANCE IN PICO FARADS PER METER
```

```
C REMARKS
C     Valid for Z > 0 0 Ohms, and EPS equal to or greater than 1 0
C     Accuracy
C     For FREE SPACE impedance between 23 3 and 1270 0
C     Ohms, the calculation exactly follows eq (3 23) with an
C     absolute accuracy of approximately 0 001*Z including
C     refraction  A warning notice is given when outside the region
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DCELI DCF,DELD DZL2 DFN2_1 DFN2_2 DPARAM DQG32 DSCD DVALV DZERO
C METHOD
C     The program computes the characteristic impedance using (3-27)
C     and effective dielectric constant using (3 23)  Iteration is
C     used to find the effective dielectric constant appropriate to
C     the desired impedance and the specified relative dielectric
C     constant of the substrate  Unbalanced microstrip calculations
C     are made using balanced line calculations and the method of
C     images
C REFERENCE
C     A  Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990
C
      IMPLICIT REAL*8 (A-H,O-Z)
      CHARACTER*1 UE
      LOGICAL IBU
      INTEGER IER
      COMMON /MIST/ XK,XKP XKK XKKP EK EKP
      COMMON /MSUB/ KEPS,TARGET
      EXTERNAL DFN2_1
C
C /MSUB/ provides communication with the function subprogram DFN2_1
C DFN2_1 is the external function subprogram called by DZERO
C
  100 IBU= TRUE
      BU=2 D0
      WRITE (5,1)
      READ (5,2) UE
      IF(UE EQ 'E' OR UE EQ 'e') THEN
        BU=1 D0
        IBU= FALSE
      ENDIF
  200 WRITE (5,3)
      READ (*,5) Z
      IF(Z EQ 0 D0) GOTO 300
      IF(Z LT 0 D0) GOTO 100
C
      WRITE (5,4)
      READ (*,5) REPS
      IF(REPS LT 1 D0) GOTO 200
C
C Initialize
C
      TOL = 1 D 10
      TARGET = Z*RD
C
C Compute the upper and lower bounds on effective dielectric constant
C
      EFFL = 0 5D0*(REPS + 1 D0)
      EFFH = REPS
      Z0L = DSQRT(EFFL)*TARGET*0 999D0
      Z0U = DSQRT(EFFH)*TARGET*1 001D0
C Check upper bound of Z
      Z0 = Z0U
      CALL DEL0(Z0)
      TEST = EKP/XKKP
      IF(TEST XK LT 1 D 8) THEN
        WRITE(5,6)
        GOTO 200
      ENDIF
C Check lower bound of Z
      Z0=Z0L
      CALL DEL0(Z0)
      IF(XK*XK LT 1 D-37) THEN
        WRITE(5,7)
        GOTO 200
      ENDIF
C
C Iterate to 2  the microstrip impedance on the dielectric substrate
C
      Z00 = DZERO(Z0L,Z0U,DFN2_1,TOL,IER)
C
C Compute the microstrip parameters
C
      EPS=REPS
      CALL DFN2_2(Z00 EPS Z,AH BH,VP CAP)
C
```

```
C Z00 is the free space impedance of the microstrip
C
C Write the header
      IF(IBU) THEN
        WRITE (5,8)
      ELSE
        WRITE (5,9)
      ENDIF
      WRITE (5 10)
C Write the calculated values
      WRITE(5,11) Z/BU BU*AM VP BU*CAP
      WRITE(5,12) Z00/BU
C
C Format statements
C
    1 FORMAT
     &(/' Unbalanced line is default  Analyze balanced? U/B [U] ',$)
    2 FORMAT(A1)
    3 FORMAT
     &(//' Type impedance  [0 0] to quit  [ 1 0] to change U/B  Z = ',$)
    4 FORMAT
     &(' Type     [ EPS  GE  1 0 ]      [0.0] for new Z      EPS = ',$)
    5 FORMAT(D25 16)
    6 FORMAT(17X,'*** Z TOO HIGH  REENTER Z ***')
    7 FORMAT(17X,'*** Z TOO LOW  REENTER Z ***')
    8 FORMAT(27X,'UNBALANCED')
    9 FORMAT(28X 'BALANCED')
   10 FORMAT(8X,'Z',14X 's/h' 13X,'vp',11X,'cap pf/m')
   11 FORMAT(4G16 6)
   12 FORMAT(21X,' Z0 = ' G20 15)
      GOTO 200
  300 STOP
      END


      SUBROUTINE DAREA(R AP,U0,CPMAX)
C PURPOSE
C     Integretes the dielectric eres, in the w plane  of generalized
C     microstrip
C USAGE
C     CALL DAREA(R,AP U0 CPMAX)
C DESCRIPTION OF PARAMETERS
C     CAPMAX   - RESULT VALUE OF DIELECTRIC AREA IN DOUBLE PRECISION
```

```
C     R          - INPUT VALUE OF THE PARAMETER R IN DOUBLE PRECISION
C     AP         - INPUT VALUE OF A'' IN DOUBLE PRECISION
C     U0         - INPUT VALUE OF U0 IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2,DSCD DFINDU DFINDV DQG32,DXOFU DXOFV DZERO
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DELO,DCONST DPARAM
C METHOD
C     Integration is by 32 point Gaussian quadrature   The dielectric
C     boundary is found by iteration  using the nonlinear real zero
C     finding routine DZERO
C REFERENCE
C     A. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation  University of Florida, Gainesville, FL
C     1990
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /NIST/ XK,XKP XKK XKKP,ER,XKP
      COMMON /AREA/ SU,CU,DU SV,CV,DV,UL,UH VL
     &             VH,0 EU TARGXH,TOL UDD,RR
      EXTERNAL DXOFV, DFINDU, DFINDV
C
C Initialize subareas and set tolerance for DZERO
C
      AREA1=0 D0
      AREA2=0 D0
      AREA3=0 D0
      TOL = 1 0 1D
      UD0=U0
      RR=R
C
C Determine whether or not the dielectric boundary X/H = 1 D.
C crosses the equipotential line U = U0
C
C Compute input parameters for routine DXOFV
C
      CALL DSCD(SU,CU,DU,U0*XKK XKP)
      X=SU/CU
      EU=DEL2(X XKP 1 D0,XKP*XKP)
C
C Set target and search range for DZERO
```

```
C
      TARGXH = 1.D0
      U = U0
      VL = AP
      VH = 0.99999999D0
      V0=0.D0
C
      V0 = DZERO(VL,VH,DXOFV,TOL,IER)
      IF(IER.EQ.1) THEN
        V0=0.99999999D0
      ENDIF
C
C Set target, integration limits UL and UH, search limits VL and VH
C
      TARGXH = 1.D0
      UL=U0
      UH=1.D0
      VL=AP
      VH=V0
C
C Integrate dielectric area below boundary X/H = 1
C
      CALL DQC32(UL,UH,DFINDV,AREA1)
      IF(R.EQ.0.D0) THEN
        AREA1=2.D0*AREA1
        GOTO 30
      ENDIF
      IF(R.EQ.1.D0) GOTO 30
C
C Set target, integration limits UL and UH, search limits VL and VH
C
      TARGXH = -1.D0
      UL=-1.D0
      UH=U0
      VL=AP
      VH=0.99999999D0
C
C Integrate dielectric area below boundary X/H = -1
C
      CALL DQC32(UL,UH,DFINDV,AREA2)
C Skip integration of AREA3 if IER=1, i.e. no inflection
      IF(IER.EQ.1) GOTO 30
C
```

```
C Compute tha area under tha inflaction af X/H = 1
C
C Set target  integration limits VL and VH, search limits UL and UH
C
      TARGXH=1 D0
      VL=U0
      VH=1 D0
      UL=-0 9999999D0
      UH=U0 + 0.0001D0
C
C Integrate tha inflaction area
C
      CALL DQG32(VL,VH,DFINDU,AREA3)
C
   30 CPMAX = 0 5D0*(AREA1+AREA2 AREA3)
      RETURN
      END

      SUBROUTINE DCELI(XK,XKP XKK E)
C PURPOSE
C     Given the moduli  k and k' computes the complete elliptic
C     integrals  K(k)  E(k)
C USAGE
C     CALL DCELI(XK XKP XKK E) to calculate K and E
C     CALL DCELI(XKP XK XKKP EP) to calculate K' and E'
C DESCRIPTION OF PARAMETERS
C     XK         INPUT VALUE OF k IN DOUBLE PRECISION
C     XKP        INPUT VALUE OF k' IN DOUBLE PRECISION
C     XKK        RESULT VALUE OF K(k) IN DOUBLE PRECISION
C     E          RESULT VALUE OF E(k) IN DOUBLE PRECISION
C     XKKP       RESULT VALUE OF K'(k) IN DOUBLE PRECISION
C     EP         RESULT VALUE OF E'(k) IN DOUBLE PRECISION
C METHOD
C     Calculation is uses the arithmetic/geometric mean
C REFERENCE
C     M  Abramowitz and I  A  Stegun  HANDBOOK OF MATHEMATICAL
C     FUNCTIONS, seventh printing  Washington D C   National Bureau
C     of Standards  pp  579 598 599, 1968
C
      REAL*8 A,AA,B BB C CS D DPI E,XK,XKP XKK
      PARAMETER(DPI=3 14159265358979323800)
C
C                COMPUTE COMPLETE ELLIPTIC INTEGRALS   K  E
```

```
C
C Initialize AGM loop
      A=1.D0
      B=XKP
      CS=XK*XK
      D=1.D0
C AGM process
    1 AA=A
      C=(A-B)*.5D0
      BB=DSQRT(A*B)
      A=(A+B)*.5D0
      D=D+D
      CS=CS+D*C*C
      IF(DABS(AA-B).LT.1.D-9*AA) GOTO 2
      B=BB
      GOTO 1
    2 XKK=.5D0*DPI/A
      R=(1.D0-.5D0*CS)*XKK
      RETURN
      END

      SUBROUTINE DCONST(R,C1,C2)
C PURPOSE
C     Computes the constants C1 and C2 for generalized microstrip
C USAGE
C     CALL DCONST(R,C1,C2)
C DESCRIPTION OF PARAMETERS
C     R       - INPUT VALUE OF R IN DOUBLE PRECISION
C     C1      - RESULT VALUE OF C1 IN DOUBLE PRECISION
C     C2      - RESULT VALUE OF C2 IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     NONE
C PREVIOUS SUBROUTINE AND FUNCTION CALLS REQUIRED
C     DELO
C METHOD
C     The constants C1 and C2 are found by the quadratic formula
C     using eqs. (A-48) and (A-49)
C REFERENCE
C     A. L. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL
C     1990
```

```
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
C
      SC0 = EKP/XKKP
      AC = 1.D0 - R*R*SC0
      BC = R*(1.D0 + XK*XK - 2.D0*SC0)
      CC = SC0 - R*R*XK*XK
      Q  = -0.5D0*(BC - DSQRT(BC*BC + 4.D0*AC*CC))
      C1 = Q/AC
      C2 = CC/Q
      RETURN
      END

      SUBROUTINE DCP(AH,BH,CP)
C PURPOSE
C     Given the conductor widths, a/h and b/h, of generalized
C     microstrip, DCP computes  CP  the true parallel dielectric
C     capacitance of generalized microstrip
C USAGE
C     CALL DCP(AH,BH,CP)
C DESCRIPTION OF PARAMETERS
C     AH - INPUT VALUE OF a/h IN DOUBLE PRECISION
C     BH - INPUT VALUE OF b/h IN DOUBLE PRECISION
C     CP - RESULT VALUE OF CP IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DCELI
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DELO,DCONST,DPARAM
C METHOD
C     DCP implements eqs. (B.6) and (B.7) of Appendix B the reference
C REFERENCE
C     A. L. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      PARAMETER (DPI=3.14159265358979323D0)
C
C Compute microstrip transmission line capacitance/epsilon0
```

```
      CMS=XKKP/XKK
C
C Compute true parallel dielectric capacitance
C
      T=.25D0*DPI
      AT=AM*T
      BT=BM*T
      IF(AM.GT.9.6D0) THEN
        IF(DEXP(-1.01D0*(BT-AT)).EQ.0.D0) THEN
          THETA=.5D0*DPI
        ELSE
          THETA=ATAN(DEXP(BT-AT))
        ENDIF
        CSL=(DLOG(16.D0)+4.D0*(AT+DLOG(DSIN(THETA))))/DPI
      ELSE
        THA=DTANH(AT)
        IF(BT.GT.14.D0) THEN
          THB=1.D0
        ELSE
          THB=DTANH(BT)
        ENDIF
        XL=THA*THB
        AA=1.D0/DCOSH(AT)
        BB=0.D0
        IF(DEXP(-1.01D0*BT).NE.0.D0) BB=1.D0/DCOSH(BT)
        XLP=AA*AA+BB*BB-AA*AA*BB*BB
        XLP=DSQRT(XLP)
        CSL=2.D0*DPI/(DLOG(16.D0)-2.D0*(DLOG(THA)+DLOG(THB)))
        CSL=CSL/CMS
        CALL DCELI(XL,XLP,XLL,E)
        CALL DCELI(XLP,XL,XLLP,EP)
        CSL=2.D0*XLL/XLLP
      ENDIF
      CP=CSL/CMS
      RETURN
      END

      SUBROUTINE DELQ(Q)
C PURPOSE
C     Given K/K' compute the modulus, k, complementary modulus, k',
C     and the complete elliptic integrals, K(k), K(k'), E(k), E(k')
C USAGE
```

```
C     CALL DELO(Z)
C DESCRIPTION OF PARAMETERS
C     Z       - INPUT VALUE OF IMPEDANCE IN DOUBLE PRECISION
C     XK      - RESULT VALUE OF k IN DOUBLE PRECISION
C     XKP     - RESULT VALUE OF k' IN DOUBLE PRECISION
C     XKK     - RESULT VALUE OF K(k) IN DOUBLE PRECISION
C     XKKP    - RESULT VALUE OF K(k') IN DOUBLE PRECISION
C     EK      - RESULT VALUE OF E(k) IN DOUBLE PRECISION
C     EKP     - RESULT VALUE OF E(k') IN DOUBLE PRECISION
C     DLNK    - RESULT VALUE OF DLOG(k) IN DOUBLE PRECISION
C     DLNKP   - RESULT VALUE OF DLOG(k') IN DOUBLE PRECISION
C REMARKS
C     Valid for K/K'>0  Output is transferred via common block /MIST/
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     NONE
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     NONE
C METHOD
C     The moduli are given by ratios of theta functions obtained by
C     evaluation of series in the nome  The arithmetic/geometric mean
C     is used to evaluate the complete elliptic integrals
C
C     For very low and very high impedances asymptotic expressions are
C     used for both the moduli and the complete elliptic integrals
C REFERENCES
C     R  Bulirsch  "Numerical calculations of elliptic integrals and
C     functions," Handbook Series of Special Functions  NUMERISCHE
C     MATHEMATIK  Vol  7  pp  78-90, 1965
C
C     M  Abramowitz and I  A  Stegun  HANDBOOK OF MATHEMATICAL
C     FUNCTIONS, seventh printing  Washington D C   National Bureau
C     of Standards  pp  597-599  1968
C
C     P  Byrd and M  Friedman  HANDBOOK OF ELLIPTIC INTEGRALS
C     FOR ENGINEERS AND SCIENTISTS  second edition 1971  pg  298
C
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER I,N
      LOGICAL FLG
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      PARAMETER(DPI=3.141592653589793238D0,DLN2=.6931471805599453D0)
      PARAMETER(DPIOV2=0.5D0*DPI)
C
```

```
      ETA = 4.D0*DPI*29.9792458D0
      RK=Z/ETA
C
C  Compute the modulus and complementary modulus by series expansion
C  of the nome
C
C  Test RK  use q if Z.GT.ETA else q'
C
      FLG = (Z.GT.ETA)
      G=RK
      IF(FLG) G = 1.D0/RK
C
C  For Z .LT. 31. .OR. Z .GT. 458D  then the approximation (B.F. 9D1.00)
C  gives the modulus within  D(1.D-13))  so bypass the series expansion
C
      IF(Z.GE.31.D0.AND.Z.LT.458.D0) THEN
C
C Initialize the series
C
        A=1.D0
        AA=DEXP(-DPI*G)
        B=1.D0
        C=1.D0
        SGN=-1.D0
C
        DO 1 I=1,15
          N=I
          N=N*N
          C=C*SGN
          D=C*AA**N
          B=B+2.D0*D
          A=A+2.D0*DABS(D)
          IF (DABS(D).LT.DABS(B)*1.D-16) GOTO 11
    1   CONTINUE
   11   XK=(B/A)*(B/A)
        XKP=DSQRT(1.D0-XK*XK)
C
C Compute the complete elliptic integrals  K, K'  E, E'  using the
C arithmetic/geometric mean
C
C Initialize the AGM loop
C       DLNK=DLOG(XK)
        DLNKP=DLOG(XKP)
```

```
         A=1 DO
         B=XKP
         CS=XK*XK
         D=1 DO
C
C Do AGM process
C
         DO 2 I = 1,15
           AA=A
           C=(A B)* 5DO
           BB=DSQRT(A*B)
           A=(A+B)* 5DO
           D=D+D
           CS=CS+D*C*C
           IF(DABS(AA B) LT 1 D 9*AA) GOTO 22
           B=BB
    2    CONTINUE
   22    XKK=DPIOV2/A
         EK=(1 DO - 5DO*CS)*XKK
         XKKP=XKK/G
C
C  Use known values of K, K' and E to compute E' using Legendre's
C  relationship
C
         EKP=DPIOV2/XKK+XKKP*(1 DO EK/XKK)
        ELSE
C
C  Use first term of B F  901 00 for moduli and asymptotic values for
C  the complete elliptic integrals
C
         DLNK=2 DO*DLN2 DPIOV2/C
         DLNKP=0 DO
         XK=DEXP(DLNK)
         XKP=1 DO
         XKK=DPIOV2
         XKKP=XKK/G
         EK=XKK
         EKP=1 DO
        ENDIF
C
C  Interchange XK, DLNK  XKK and EK with XKP  DLNKP  XKKP and EKP
C  if G=1/BK
C
```

```
      IF(FLG) THEN
        T=XKK
        XKK=XKKP
        XKKP=T
        T=EKP
        EKP=EK
        EK=T
        T=XKP
        XKP=XK
        XK=T
        T=DLNKP
        DLNKP=DLNK
        DLNK=T
      ENDIF
      RETURN
      END
```

Function Subroutine DEL2 computes the incomplete elliptic integrals of the first and second kind. It is a double precision version of the function subprogram EL2 from W H Press, B P Flannery, S A Teukolsky and W T Vetterling, "Numerical Recipes," fourth printing, Cambridge, Cambridge University Press, Pg. 186, 1987. For conversion change all constants and intrinsic functions to double precision. Insert as the first three lines

```
      REAL*8 FUNCTION DEL2(X,QQC,AA,BB)
      IMPLICIT REAL*8 (A-H,O-Z)
      PARAMETER(PI = 3.141592653589793238D0, CA = 1.D-9, CB = 1.D-17)

      REAL*8 FUNCTION DFINDU(V)
C PURPOSE
C     Provides the interface between the Gaussian quadrature routine
C     BQG32 and the iterative procedure (DZERO) for finding U(V)
C USAGE
C     U=DFINDU(V)
C DESCRIPTION OF PARAMETERS
C     V          INPUT VALUE OF V IN DOUBLE PRECISION
C     DFINDU   - RESULT VALUE OF U(V) IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2,DSCD,DXOFU,DZERO
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DELO,DCONST,DPARAM,DAREA,BQG32
C METHOD
```

```
C     DFINDU is called from DQG32 and returns U(V), which is found
C     iteratively
C REFERENCES
C     A  Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville  FL,
C     1990
C
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER IER
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      COMMON /AREA/ SU,CU,DU,SV,CV,DV,UL,UH,VL,
     &              VH,U,EU,TARGXH,TOL,U0,R
      EXTERNAL DXOFU
C
      CALL DSCD(SV,CV,DV,V*XKKP,XK)
      DFINDU=U0-DZERO(UL,UH,DXOFU,TOL,IER)
      IF(IER.NE.0) THEN
        PAUSE 'ERROR IN DFINDU'
      ENDIF
      RETURN
      END

      REAL*8 FUNCTION DFINDV(U)
C PURPOSE
C     Provides the interface between the Gaussian quadrature routine
C     DQG32 and the iterative procedure (DZERO) for finding V(U)
C USAGE
C     V=DFINDU(U)
C DESCRIPTION OF PARAMETERS
C     U       - INPUT VALUE OF U IN DOUBLE PRECISION
C     DFINDV    RESULT VALUE OF V(U) IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2,DSCD,DXOFV,DZERO
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DEL0,DCONST,DPARAM,DAREA,DQG32
C METHOD
C     DFINDV is called from DQG32 and returns V(U), which is found
C     iteratively
C REFERENCES
C     A  Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville  FL,
C     1990
C
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER IER
      COMMON /MIST/ ER,XKP,XKK,XKKP,EK,EXP
      COMMON /AREA/ SU,CU,DU,SV,CV,DV,UL,UH,VL,
     &              VH,UU,EU,TARGXH,TOL,UO,R
      EXTERNAL DXOFV
C
      UU=U
      CALL DSCD(SU,CU,DU,U*XKK,XKP)
      X=SU/CU
      EU=DEL2(X,XKP,1.D0,XKP*XKP)
      DFINDV=DZERO(VL,VH,DXOFV,TOL,IER)
      IF(IER.NE.0) THEN
        PAUSE 'Error finding V - ROUTINE DFINDV'
      ENDIF
      RETURN
      END

      REAL*8 FUNCTION DFN2_1(Z1)
C PURPOSE
C     This program is an executive function program. It is
C     substantially an implementation of MIST1 in the form of a
C     function program to allow iteration to a desired value of
C     Z on a dielectric sheet. The program computes the function
C     F(Z1) = Z - TARGET for use with DZERO in finding F(U) = 0.0
C USAGE
C     F(U)=DFN2_1(Z1)
C DESCRIPTION OF PARAMETERS
C     Z1        - INPUT VALUE OF IMPEDANCE IN DOUBLE PRECISION
C     DFN2_1    RESULT VALUE OF F(Z1) IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DCELI, DCP, DEL0, DEL2, DPARAM, DQG32, DSCD, DVALV, DZERO
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     NONE
C METHOD
C     This program is valid for R=0.D. It computes the parameters of
C     unbalanced microstrip by the method of images from the balanced
C     microstrip parameters. CPMAX is computed using the direct
C     inverse transformation DVALV to compute the area integral
```

```
C     This is much more efficient and faster than the general
C     calculation  with DAREA etc.  used in MIST1
C REFERENCE
C     A. Holloway  "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      COMMON/MSUB/REPS,TARGET
      EXTERNAL DVALV
C Calculate the moduli and complete elliptic integrals.
      Z0=Z1
      CALL DELO(Z0)
C Calculate the conductor widths a/h = b/h
      R=0.D0
      C1=DSQRT(EKP/XKKP)
      C2=C1
      CALL DPARAM(R,C1,C2,AP,BP,U0,AH,BH)
C Calculate the true parallel dielectric capacitance CP
      CALL DCP(AH,BH,CP)
C Exact formulation of external capacitance, eq. (4.16), including
C integration of dielectric area to obtain AREA-CPMAX
      CALL DQG32(0.D0,1.D0,DVALV,AREA)
      EFFK=CP*(REPS-1.D0)+AP+(1.D0-AP)/
     &      (1.D0-(1.D0-1.D0/REPS)*(AREA-CP)/(1.D0-AP))
C Compute impedance on the dielectric sheet
      Z=Z0/DSQRT(EFFK)
      DFN2_1=Z-TARGET
      RETURN
      END

      SUBROUTINE DFN2_2(Z0,REPS,Z,AH,BH,VP,CAP)
C PURPOSE
C     Given Z0 and REPS, this program computes the microstrip parameters
C     for balanced microstrip
C USAGE
C     CALL DFN2_2(Z0,REPS,Z,AH,BH,VP,CAP)
C DESCRIPTION OF PARAMETERS
C     Z0   - MICROSTRIP IMPEDANCE IN FREE SPACE
C     Z    - MICROSTRIP IMPEDANCE ON A DIELECTRIC SHEET
C     REPS - RELATIVE DIELECTRIC CONSTANT OF THE SHEET.
C     AH=a/h- RATIO OF NARROW CONDUCTOR WIDTH TO SEPARATION
```

```
C     BH=b/h   RATIO OF WIDE CONDUCTOR WIDTH TO SEPARATION
C     VP     PHASE VELOCITY RELATIVE TO THE FREE SPACE VELOCITY OF
C            LIGHT
C     CAP  - CAPACITANCE IN PICO FARADS PER METER
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DCELI, DCP  DELO  DEL2  DPARAM  DQG32  DSCO  DVALV  DZERO
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     NONE
C METHOD
C     This program is valid for R=0 0  It computes the parameters of
C     unbalanced microstrip by the method of images from the balanced
C     microstrip parameters   CPMAX is computed using the direct
C     inverse transformation DVALV to compute the area integral
C     This is much more efficient and faster than the general
C     calculation, with DAREA etc   used in MIST1
C REFERENCE
C     A  Holloway, "Analysis and Design of Generalized Microstrip."
C     doctoral dissertation  University of Florida  Gainesville  FL,
C     1990
C
      IMPLICIT REAL*8 (A-H O-Z)
      COMMON /MIST/ XK XKP XKK XKKP EK,EKP
      PARAMETER(CC=2 99792458D0)
      EXTERNAL DVALV
C
C Calculate the moduli and complete elliptic integrals
      CALL DELO(ZO)
C Calculate the conductor widths a/h = b/h
      R=0 D0
      C1=DSQRT(EKP/XKKP)
      C2=C1
      CALL DPARAM(R C1,C2,AP BP DO AM BH)
C Calculate the true parallel dielectric capacitance CP
      CALL DCP(AM,BH CP)
C Exact formulation of external capacitance  eq  (4-16)  including
C integration of dielectric area to obtain AREA=CPMAX
      CALL DQG32(0 D0 1 D0,DVALV,AREA)
      CI=CP*(REPS-1 D0)+AP
      CE=(1 D0 AP)/
     &   (1 D0 (1 D0 1 D0/REPS)*(AREA CP)/(1 D0-AP))
      EFFK=CI+CE
```

```
C Compute relative phase velocity
      VP=1.D0/DSQRT(EFFK)
C Compute impedance on the dielectric sheet
      Z=Z0*VP
C Compute capacitance per meter (pico Farads)
      CAP=1.D+4/(CC*VP*Z)
      RETURN
      END

      SUBROUTINE DPARAM(B,C1,C2,AP,BP,U0,AH,BH)
C PURPOSE
C     Computes the generalized microstrip parameters A' B' a/h, b/h
C     and U0
C USAGE
C     CALL DPARAM(B,C1,C2,AP,BP,U0,AH,BH)
C DESCRIPTION OF PARAMETERS
C     B     - INPUT VALUE OF B IN DOUBLE PRECISION
C     AP    - RESULT VALUE OF A' IN DOUBLE PRECISION
C     BP    - RESULT VALUE OF B' IN DOUBLE PRECISION
C     AH    - RESULT VALUE OF a/h IN DOUBLE PRECISION
C     BH    - RESULT VALUE OF b/h IN DOUBLE PRECISION
C     U0    - RESULT VALUE OF U0 IN DOUBLE PRECISION
C REMARKS
C     All parameters are normalized to a unit square
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DEL0,DCONST
C METHOD
C     The parameter A' is found using (A-97) with A = a'/K', B' using
C     (A-100) with B' = b'/K' AH = a/h using (A-98), BH = b/h using
C     (A-101), and U0=u0/K using (A-110)
C REFERENCE
C     A. L. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990, Appendix A, Table A.3.
C-----------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      PARAMETER(PIINV = 0.31830988618379067D0)
C
      SXK = XK*XK
C
```

```
C                    COMPUTE D0,AF,BF A/H AND B/H
C
C Compute A/H & AF
C
        X = DSQRT((C1*C1 - SXK)/(1.D0 - C1*C1))/XK
        E = DEL2(X,XK,1.D0,SXK)
        F = DEL2(X,XK,1.D0,1.D0)
        AF = F/XKKP
        ZETA = EKP*AF - E
        B0 = DSQRT((C1*C1 - SXK)*(1.D0 - C1*C1))
        SG = SG*(1.D0/C1 - R/(1.D0 + C1*R))
        AH = (SG + ZETA)*2.D0*XKKP*FIINV
C
C  Case R = 0.D0, B/H = A/H
C
      IF(R.EQ.0.D0) THEN
        BH = AH
        D0 = 0.D0
        BF = AF
      ELSEIF(R.EQ.1.D0) THEN
C
C  Case R = 1.D0, B/H-->INFINITY
C
        BH = 1.E37
        U0 = -2.D0
        BF = 1.D0
      ELSE
C
C  Compute B/H, BF
C
        X = DSQRT((C2*C2 - SXK)/(1.D0 - C2*C2))/XK
        E = DEL2(X,XK,1.D0,SXK)
        F = DEL2(X,XK,1.D0,1.D0)
        BF = F/XKKP
        ZETA = E - EKP*BF
        SH = DSQRT((C2*C2 - SXK)*(1.D0 - C2*C2))
        SH = SH*(1.D0/C2 + R/(1.D0 - C2*R))
        BH = (SH + ZETA)*2.D0*XKKP*FIINV
C
C  Compute U0
C
        X = R/DSQRT(1.D0 - R*R)
        F = DEL2(X,XKP,1.D0,1.D0)
      U0 = -F/XXX
     ENDIF
     RETURN
     END
```

SUBROUTINE DQG32 is from the IBM System/360 Scientific Subroutine Package, Version III, Programmers Manual, Program Number 360A-CM 03X, <u>Fifth Edition</u> (August 1970), pg 303. No modification of this program is required.

```
      SUBROUTINE DSCD (SN CN DN,X,CK)
C PURPOSE
C    Computes the three Jacobian elliptic functions sn  cn  dn
C USAGE
C    CALL DSCD(SN,CN DN,X CK)
C DESCRIPTION OF PARAMETERS
C   SN    - RESULT VALUE OF SN(X) IN DOUBLE PRECISION
C   CN    - RESULT VALUE OF CN(X) IN DOUBLE PRECISION
C   DN      RESULT VALUE OF DN(X) IN DOUBLE PRECISION
C   X     - DOUBLE PRECISION ARGUMENT OF JACOBIAN ELLIPTIC FUNCTIONS
C   CK    - COMPLEMENTARY MODULUS IN DOUBLE PRECISION
C REMARKS
C   This subroutine avoids repeated calculation of the AGM for
C   the same value of CK when repeated calls are made using the
C   same value of CK with many values of X
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C   NONE
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C   NONE
C METHOD
C   Calculation is done using the process of the arithmetic/
C   geometric mean together with Gauss' descending transformation
C   before Inversion of the integral takes place
C REFERENCE
C   R  Bulirsch, NUMERICAL CALCULATION OF THE ELLIPTIC INTEGRALS
C   AND ELLIPTIC FUNCTIONS
C   HANDBOOK SERIES OF SPECIAL FUNCTIONS
C   NUMERISCHE MATHEMATIK VOL  7, 1965, PP. 78-90
C
      DIMENSION EM(13) EN(13)
      REAL*8 A B,C CC CK,CN,D DN,EM EMC EN,OLD,SN U X
      INTEGER I,II,L
      LOGICAL FLG
C
```

```
      EMC=CK*CK
      U=X
      IF(EMC.NE.0.D0)THEN
        FLG=(EMC.LT.0.D0)
C Test for imaginary modulus
        IF(FLG) THEN
C Jacobi's imaginary modulus transformation
          D=1.D0-EMC
          EMC=-EMC/D
          D=DSQRT(D)
          U=U*D
        ENDIF
        DN=1.D0
C Do AGM process only if CK has changed.
        IF(CK.NE.OLD) THEN
          A=1.D0
          DO 1 I=1,13
            L=I
            EM(I)=A
            EMC=DSQRT(EMC)
            EN(I)=EMC
            CC=0.5D0*(A+EMC)
            IF(DABS(A-EMC).LE.1.D-9*A) GOTO 11
            EMC=A*EMC
            A=CC
1         CONTINUE
        ENDIF
C Go to backward recursion if EMC has not changed
11      C=CC
        U=C*U
        SN=DSIN(U)
        CN=DCOS(U)
        IF(SN.EQ.0.D0) GOTO 22
        A=CN/SN
        C=A*C
        DO 2 II=L,1,-1
          B=EM(II)
          A=C*A
          C=DN*C
          DN=(EN(II)+A)/(B+A)
          A=C/B
2       CONTINUE
        A=1.D0/DSQRT(C**2+1.D0)
        IF(SN.LT.0.D0)THEN
          SN=-A
        ELSE
          SN=A
        ENDIF
        CN=C*SN
C Test for imaginary modulus
22      IF(FLG)THEN
C Finish Jacobi's imaginary transformation
          A=DN
          DN=CN
          CN=A
          SN=SN/D
        ENDIF
C Approximation for CK near 0 D0, (k near 1 D0)
      ELSE
        CN=1.D0/DCOSH(U)
        DN=CN
        SN=DTANH(U)
      ENDIF
      OLD=CK
      RETURN
      END


      REAL*8 FUNCTION DVALV(U)
C PURPOSE
C     Given U this function subroutine computes the ordinate V(U) for
C     X=x/h=1.D0  in connection with integration of the dielectric
C     area
C USAGE
C     V=DVALV(U)
C DESCRIPTION OF PARAMETERS
C     U        - DOUBLE PRECISION VALUE OF U=u/K'
C     V        - RESULTING VALUE OF V=v/K' IN DOUBLE PRECISION
C REMARKS
C     DVALV is called by DQG32
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2 DSCD
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DELG
C METHOD
C     DVALV implements the inverse transform functions (A-70) (A-71)
```

```
C     (A-72), and (A-73) of Table A-3 of the reference
C REFERENCE
C     A. L. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
C
      CALL DSCD(SU,CU,DU,U*XKK,XKP)
      X=SU/CU
      E=DEL2(X,XKP,1.D0,XKP*XKP)
      FACT=.5D0*3.14159265358979320D0/XKKP*(1.D0-U)+U*EK-E
      X=DSQRT(FACT/(SU*(CU*DU-SU*FACT)))/XK
      F=DEL2(X,XK,1.D0,1.D0)
      DVALV=F/XKKP
      RETURN
      END

      REAL*8 FUNCTION DXOFU(U)
C PURPOSE
C     Find the value of F(U) = TARGXH - x/h corresponding to the
C     w-plane coordinate pair (U,V)
C USAGE
C     F(U) = DXOFU(U)
C DESCRIPTION OF PARAMETERS
C     U        - INPUT VALUE OF U IN DOUBLE PRECISION
C     DXOFU    - RESULT VALUE OF F(U) IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2, DSCD
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     DELO, DCONST, DPARAM, DZERO
C METHOD
C     Implements equation (A-64) with V held constant.  This function
C     program is used in conjunction with DZERO to find F(U) = 0.D
C REFERENCE
C     A. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL,
C     1990
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /MIST/ XK,XKP XKK XKKP EK EKP
      COMMON /AREA/ SU CU,DU,SV,CV DV UL UH,VL
     &              VH,UU,EU,TARGXH,TOL,UD,R
C
      XH=0 D0
      CALL DSCD(SU CU DU,U*XKK,XKP)
C
      CV1= CV/XK
      DV1= DV/XK
      CV2= CV1*CV1
      SV2= SV*SV
      T1= SU*SU*SV2
      T2= CV2+T1
      T3= SU*CU*DU
      IF(R NE 0 D0) THEN
        DV2= DV1*DV1
        ACF= T3*DV2*(CV2-T1)/T2
        BDF= T3*SV2*CV2*(CU*CU + DU*DU*DV2)/T2
        AAK= R*CU*DU*DV1*(CV2 - T1)
        DEN= (R*SU*DV1 + T2)**2 + (R*CU*SU*SV*CV1)**2
        XH= (R**2*(ACF-BDF)+AAK)/DEN
      ENDIF
      EF= T3*SV2/T2
      X=SU/CU
      E=DEL2(X XKP 1 D0,XKP*XKP)
      XH= U + 2 D0*XKKP* 3183098861837907D0*(E-EK*U+XH+EF)
      DXOFU=TARGXH-XH
      RETURN
      END

      REAL*8 FUNCTION DXOFV(V)
C PURPOSE
C     Find the value of F(V) = TARGXH   x/h corresponding to the
C     w plane coordinate pair (U V)
C USAGE
C     F(V) = DXOFV(V)
C DESCRIPTION OF PARAMETERS
C     V         INPUT VALUE OF V IN DOUBLE PRECISION
C     DXOFV     RESULT VALUE OF F(V) IN DOUBLE PRECISION
C REMARKS
C     NONE
C SUBROUTINE AND FUNCTION PROGRAMS REQUIRED
C     DEL2  DSCD
```

```
C PREVIOUS SUBROUTINE AND FUNCTION PROGRAM CALLS REQUIRED
C     BELO, DCONST, DPARAM, DZERO
C METHOD
C     Implements equation (A-64) with U held constant.  This function
C     program is used in conjunction with DZERO to find F(V) = 0.0
C REFERENCE
C     A. Holloway, "Analysis and Design of Generalized Microstrip,"
C     doctoral dissertation, University of Florida, Gainesville, FL
C     1990
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON /MIST/ XK,XKP,XKK,XKKP,EK,EKP
      COMMON /AREA/ SU,CU,DU,SV,CV,DV,UL,UH,VL,
     &              VH,U,EU,TARGXH,TOL,UO,R
C
      XH=0.D0
      CALL DSCD(SV,CV,DV,V*XKKP,XK)
      CV1=CV/XK
      DV1=DV/XK
      CV2= CV1*CV1
      SV2= SV*SV
      T1= SU*SU*SV2
      T2= CV2 + T1
      T3= SU*CU*DU
      IF(R.NE.0.D0) THEN
        DV2= DV1*DV1
        ACF= T3*DV2*(CV2 - T1)/T2
        BDF= T3*SV2*CV2*(CU*CU + DU*DU*DV2)/T2
        AAK= R*CU*DU*DV1*(CV2 - T1)
        DEN= (R*SU*DV1 + T2)**2 + (R*CU*DU*SV*CV1)**2
        XH= (R**2*(ACF-BDF)+AAK)/DEN
      ENDIF
      EF= T3*SV2/T2
      XH= U + 2.D0*XKKP*.31830988618379067D0*(SU-EK*U+XH+EF)
      DXOFV=TARGXH-XH
      RETURN
      END
```

Subroutine DZERO is a program to find the real zeros of nonlinear functions within a range. DZERO is a slightly modified double precision version of the subprogram ZEROIN from G. E. Forsythe, M. A. Malcolm and C. B. Moler, "Computer Methods for Mathematical Computations," Englewood Cliffs, NJ: Prentice Hall, Inc., pp. 164-166, 1977. Convert all constants and intrinsic functions to double precision The first four lines should read

```
      REAL*8 FUNCTION DZERO(AX,BX,F,TOL,IER)
      REAL*8 AX,BX,F,TOL
      REAL*8 A,B,C,D,E,EPS,FA,FB,FC,TOL1,XM,P,Q,R,S
      INTEGER IER
```

Add the IF statement as shown

```
C
C BEGIN STEP
C
      IF(DSIGN(1.D0,FA)+DSIGN(1.D0,FB)) 100,20,100
   20 C = A
```

Change the last lines to read

```
C
C DONE
C
   90 DZERO = B
      RETURN
  100 IER = 1
      RETURN
      END
```

These changes do not interfere with the normal operation of the program and are required for proper program function with subroutine DAREA

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BIOGRAPHICAL SKETCH

Albert L. Holloway was born in Boynton Beach, FL, in 1927 He received the B E.E. degree (with high honors) in 1959 and the M E.E. degree in 1962, both from the University of Florida at Gainesville

He is a member of Tau Beta Pi, Phi Kappa Phi and Pi MU Epsilon He holds several U.S. patents in the field of antenna technology and has published in the fields of microwave and antenna technology.

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